§1 Power Series Expansions

1.1 Weierstrass's Theorem

Theorem 1. (Weierstrass's Theorem) Consider the sequence $\{f_n\}_{n=1}^{\infty}$, where f_n is analytic on the open connected set Ω_n. Suppose in addition that

$$\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \cdots \quad \text{and that} \quad \bigcup_{n=1}^{\infty} \Omega_n = \Omega.$$

If $\{f_n\}_{n=1}^{\infty}$ converges to a limit function f in the open connected set Ω, uniformly on every compact subset of Ω, then f is analytic in Ω.

Moreover, f_n' converges uniformly to f' on every compact subset of Ω.

Proof Let E be a compact subset of $\Omega = \cup_{n=1}^{\infty}\Omega_n$, and let

- $N \in \mathbb{N}$ be chosen such that $E \subset \Omega_n$ for all $n \geq N$.
- $z_0 \in E \subset \Omega$ and $R > 0$ (depending on z_0) be chosen such that $\overline{D}_R(z_0) \subset \Omega_n$ for all $n \geq N$.

By Cauchy's integral formula, for all $n \in \mathbb{N}$ such that $n \geq N$, we have

$$f_n(z) = \frac{1}{2\pi i}\int_{\partial \overline{D}_R(z_0)} \frac{f_n(\zeta)}{\zeta - z}\, d\zeta \quad , \quad \forall\, z \in D_R(z_0)$$

Since f_n converges uniformly to f on $\overline{D}_R(z_0)$, we have

$$f(z) = \lim_{n\to\infty} \frac{1}{2\pi i}\int_{\partial \overline{D}_R(z_0)} \frac{f_n(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i}\int_{\partial \overline{D}_R(z_0)} \frac{\lim_{n\to\infty} f_n(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i}\int_{\partial \overline{D}_R(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

for all $z \in D_R(z_0)$, which implies that f is analytic in the disk and from which we conclude that f is analytic in Ω.

Furthermore,

$$f_n'(z) = \frac{1}{2\pi i}\int_{\partial \overline{D}_R(z_0)} \frac{f_n(\zeta)}{(\zeta - z)^2}\, d\zeta \quad , \quad \forall\, z \in D_R(z_0)$$

so

$$\lim_{n\to\infty} f_n'(z) = \frac{1}{2\pi i}\int_{\partial \overline{D}_R(z_0)} \frac{\lim_{n\to\infty} f_n(\zeta)}{(\zeta - z)^2}\, d\zeta = \frac{1}{2\pi i}\int_{\partial \overline{D}_R(z_0)} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta = f'(z)$$

for all $z \in D_R(z_0)$. For each $z \in \overline{D}_{R/2}(z_0)$, since

$$|f_n'(z) - f'(z)| \leq \frac{1}{2\pi}\int_{\partial \overline{D}_R(z_0)} \frac{|f_n(\zeta) - f(\zeta)|}{|\zeta - z|^2}\, |d\zeta| \leq \frac{1}{2\pi}\int_{\partial \overline{D}_R(z_0)} \frac{|f_n(\zeta) - f(\zeta)|}{(R/2)^2}\, |d\zeta|$$

for all $n \in \mathbb{N}$ such that $n \geq N$, and the fact that E can be covered by a finite number of closed disks in $\{\overline{D}_{R/2}(z_0) \mid z_0 \in E,\ R = R(z_0)\}$, the convergence of f_n' to f' is uniform on E.

Similarly, one can show that $f_n^{(k)}$ converges uniformly to $f^{(k)}$ on every compact subset of Ω for all $k \in \mathbb{N}$.

Theorem 2. (Hurwitz's Theorem) If the functions f_n are analytic and nowhere zero in an open connected set Ω, and if f_n converges to f uniformly on every compact subset of Ω, then f is either identically zero, or never equal to zero in Ω.

Proof : Suppose there exists $z_0 \in \Omega$ such that $f(z_0) = 0$ and that f is not identically zero. Since f is analytic, $\exists\, \delta > 0$ such that $\forall\, z \in \overline{D}_\delta(z_0) \setminus \{z_0\} \subset \Omega$, $f(z) \neq 0$. On $\partial \overline{D}_\delta(z_0)$, since

$$\lim_{n\to\infty} \frac{1}{f_n} = \frac{1}{f} \quad \text{and} \quad \lim_{n\to\infty} f_n' = f'$$

both uniformly, we may thus write

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_{\partial \overline{D}_\delta(z_0)} \frac{f_n'(z)}{f_n(z)}\, dz = \frac{1}{2\pi i} \int_{\partial \overline{D}_\delta(z_0)} \frac{f'(z)}{f(z)}\, dz$$

By the argument principle, the left-hand side is 0, and the equality cannot hold since $f(z_0) = 0$. There is a contradiction. This concludes our proof.

1.2 The Taylor Series

Corollary Let $\{f_n\}_{n=1}^\infty$ be a sequence of analytic functions on an open connected set Ω. If the infinite series

$$\sum_{n=1}^{\infty} f_n(z) = \lim_{N\to\infty} \sum_{n=1}^{N} f_n(z) = f(z)$$

converges uniformly to f on every compact subset of Ω, then f is analytic in Ω and the series can be differentiated term by term.

Proof Let $g_n = \displaystyle\sum_{i=1}^{n} f_i$, and apply the Weierstrass' Theorem for the sequence $(g_n)_{n=1}^\infty$.

Theorem 3. If $f(z)$ is analytic in an open connected set Ω containing z_0, then the representation

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k\,!} (z - z_0)^k$$

is valid in the largest open disk of center z_0 contained in Ω.

Proof For each $z_0 \in \Omega$, we know from Section 4.3.1 that we can write the Taylor formula of f as

$$f(z) = \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k\,!} (z - z_0)^k + f_{n+1}(z)\,(z - z_0)^{n+1}$$

where

$$f_{n+1}(z) = \frac{1}{2\pi i} \int_{\partial \overline{D}} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}\,(\zeta - z)}\, d\zeta$$

and $\overline{D}$ is any closed disk $|z - z_0| \leq R$ contained in Ω.

Let M be the maximum of $|f|$ on $\partial \overline{D}$. Since

$$\begin{aligned} &\left| f_{n+1}(z)\,(z - z_0)^{n+1} \right| \leq \frac{M |z - z_0|^{n+1}}{R^n\,(R - |z - z_0|)} \quad \forall\, |z - z_0| < R \\ \Longrightarrow\ &\lim_{n\to\infty} \left| f_{n+1}(z)\,(z - z_0)^{n+1} \right| = 0 \quad \text{uniformly in every disk } |z - z_0| \leq r < R, \end{aligned}$$

This proves the existence of a Taylor series for f centered in z_0 which is valid in the largest open disk of center z_0 contained in Ω.

1.3 The Laurent Series

We say that $\sum_{k=-\infty}^{\infty} \mu_k = L$ if both $\sum_{k=0}^{\infty} \mu_k$ and $\sum_{k=1}^{\infty} \mu_{-k}$ converge and if the sum of their sums is L.

Definition A Laurent expansion of a function $f(z)$ about an isolated singularity α is a series of the form

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z-\alpha)^k.$$

Proposition $f(z) = \sum_{k=-\infty}^{\infty} c_k z^k$ is convergent in the domain

$$D = \{z \in \mathbb{C} \mid R_1 < |z| \text{ and } |z| < R_2\}$$

where

$$\frac{1}{\limsup_{k\to\infty} |c_k|^{1/k}} \geq R_2 \quad \text{and} \quad \limsup_{k\to\infty} |c_{-k}|^{1/k} \leq R_1.$$

If $R_1 < R_2$, $D = A(R_1, R_2) = \{z \in \mathbb{C} \mid R_1 < |z| < R_2\}$ is an annulus and f is analytic in D.

Proof Write $f(z) = f_1(z) + f_2(z)$, where

$$f_1(z) = \sum_{k=0}^{\infty} c_k z^k \quad \text{and} \quad f_2(z) = \sum_{k=1}^{\infty} c_{-k} \left(1/z\right)^k.$$

Since

- $f_1(z) = \sum_{k=0}^{\infty} c_k z^k$ converges for $|z| < \dfrac{1}{\limsup_{k\to\infty} |c_k|^{1/k}}$,
- $f_2(z) = \sum_{k=1}^{\infty} c_{-k} \left(1/z\right)^k$ converges for $\dfrac{1}{|z|} < \dfrac{1}{\limsup_{k\to\infty} |c_{-k}|^{1/k}} \iff |z| > \limsup_{k\to\infty} |c_{-k}|^{1/k}$,

the sum $\sum_{k=-\infty}^{\infty} c_k z^k$ converges for all $z \in D$.

Also, since f_1 is a power series and $f_2(z) = g(1/z)$ where g is a power series, f_1 and f_2 are both analytic in their respective domains of convergence. Hence f is analytic in the intersection $A(R_1, R_2)$ of their domains. $\square$

Theorem If $f(z)$ is analytic in the annulus $A(R_1, R_2) = \{z \in \mathbb{C} \mid R_1 < |z| < R_2\}$, then f has a Laurent expansion

$$f(z) = \sum_{k=-\infty}^{\infty} c_k z^k \quad \text{converges for all } z \in A(R_1, R_2),$$

where

$$c_k = \frac{1}{2\pi i} \int_{C_\rho(0)} \frac{f(w)}{w^{k+1}}\, dw$$

for each $k \in \mathbb{Z}$, and for any circle $C_\rho(0)$ of radius $R_1 < \rho < R_2$ with center at 0.

Proof For each $z \in A(R_1, R_2)$, consider the function $g : A(R_1, R_2) \to \mathbb{C}$ defined by

$$g(w) = \begin{cases} \dfrac{f(w) - f(z)}{w - z} & \text{if } w \neq z, \\ f'(z) & \text{if } w = z. \end{cases}$$

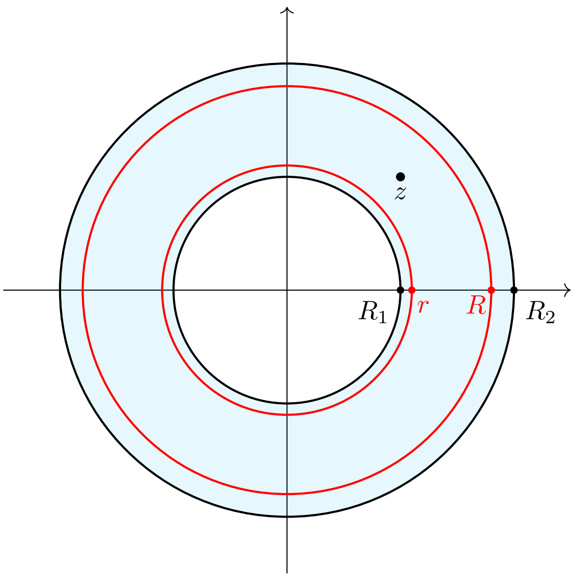
Since g is analytic in $A(R_1, R_2) \setminus \{z\}$, and continuous in $A(R_1, R_2)$, so g satisfies the condition $\lim_{w \to z} (w - z) g(w) = 0$, and, by the Cauchy's Theorem (Theorem 15), we have

$$\int_\gamma g(w)\, dw = 0 \quad \text{for every cycle } \gamma \sim 0 \text{ in } A(R_1, R_2) \setminus \{z\}$$

and, by the Fundamental Theorem of Calculus, there exists an analytic function $G(w)$, an antiderivative of g, such that

$$\begin{aligned} & G'(w) = g(w) \quad \text{for each } w \in A(R_1, R_2) \\ \Longrightarrow & \int_C g(w)\, dw = \int_C G'(w)\, dw = 0 \quad \text{for any closed curve } C \subset A(R_1, R_2). \end{aligned}$$

For any r, R such that $R_1 < r < |z| < R < R_2$, let $C_R(0) = \{w = Re^{i\theta} \mid 0 \leq \theta \leq 2\pi\}$ $C_r(0) = \{w = re^{i\theta} \mid 0 \leq \theta \leq 2\pi\}$ be the positively oriented circles, and I be the line segment from $z = R$ to $z = r$ in $A(R_1, R_2)$.



Note that $C_R(0) \cup I \cup (-C_r(0)) \cup (-I)$ is a closed curve in $A(R_1, R_2)$, and

- $0 = \dfrac{1}{2\pi i} \displaystyle\int_{C_R(0) \cup I \cup (-C_r(0)) \cup (-I)} g(w)\, dw = \dfrac{1}{2\pi i} \int_{C_R(0) \cup (-C_r(0))} \dfrac{f(w) - f(z)}{w - z}\, dw,$

$$\Longrightarrow f(z) = \frac{1}{2\pi i} \int_{C_R(0) \cup (-C_r(0))} \frac{f(z)}{w - z}\, dw = \frac{1}{2\pi i} \int_{C_R(0) \cup (-C_r(0))} \frac{f(w)}{w - z}\, dw$$

- on $C_R(0)$, since $0 < r < |z| < |w| \ \forall\, w \in C_R(0) \Longrightarrow \dfrac{|z|}{|w|} < 1$, the sum

$$\sum_{k=0}^{\infty} \frac{z^k}{w^{k+1}} = \frac{1}{w} \cdot \frac{1}{1 - (z/w)} = \frac{1}{w - z} \quad \text{converges uniformly on } C_R(0),$$

- on $C_r(0)$, since $0 < |w| < |z|\ \forall\, w \in C_r(0) \implies \dfrac{|w|}{|z|} < 1$, the sum

$$-\sum_{k=-\infty}^{-1} \frac{z^k}{w^{k+1}} = -\sum_{k=0}^{\infty} \frac{w^k}{z^{k+1}} = -\frac{1}{z} \cdot \frac{1}{1-(w/z)} = \frac{-1}{z-w} \quad \text{converges uniformly on } C_r(0).$$

Hence we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C_R(0)} \frac{f(w)}{w-z}\, dw - \frac{1}{2\pi i} \int_{C_r(0))} \frac{f(w)}{w-z}\, dw \\ &= \frac{1}{2\pi i} \int_{C_R(0)} f(w) \sum_{k=0}^{\infty} \frac{z^k}{w^{k+1}}\, dw + \frac{1}{2\pi i} \int_{C_r(0)} f(w) \sum_{k=-\infty}^{-1} \frac{z^k}{w^{k+1}}\, dw \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_R(0)} \frac{f(w)}{w^{k+1}}\, dw \right) z^k + \sum_{k=-\infty}^{-1} \left(\frac{1}{2\pi i} \int_{C_r(0)} \frac{f(w)}{w^{k+1}}\, dw \right) z^k \end{aligned}$$

converges for all $z \in A(R_1, R_2)$.

Examples

(a) The following Laurent expansion converges on $A(0, \infty)$.

$$e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k!\, z^k} = 1 + \frac{1}{z} + \frac{1}{2\, z^2} + \frac{1}{3!\, z^3} + \cdots.$$

(b) On $|z - 0| < 1$, since $\dfrac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 - \cdots$, and $\dfrac{1}{(1+z)^2} = -\dfrac{d}{dz}\left(\dfrac{1}{1+z}\right)$,

$$\frac{1}{z(1+z)^2} = \frac{1}{z} \frac{1}{(1+z)^2} = \frac{1}{z}\left(1 - 2z + 3z^2 - 4z^3 + \cdots\right) = \sum_{k=0}^{\infty} (-1)^k (k+1)\, z^{k-1}$$

On $|z - (-1)| = |z+1| < 1$,

$$\begin{aligned} \frac{1}{z(1+z)^2} &= -\frac{1}{1-(z+1)} \frac{1}{(z+1)^2} \\ &= -\frac{1}{(z+1)^2}\left[1 + (z+1) + (z+1)^2 + (z+1)^3 + \cdots\right] = -\sum_{k=-2}^{\infty} (z+1)^k \end{aligned}$$

Theorem If $f(z)$ is analytic on the annulus $R_1 < |z - \alpha| < R_2$, then f has a unique representation

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z-\alpha)^k \quad \text{where } c_k = \frac{1}{2\pi i} \int_{C_\rho(\alpha)} \frac{f(z)}{(z-\alpha)^{k+1}}\, dz$$

for each $k \in \mathbb{Z}$, and for any circle $C_\rho(\alpha)$ of radius $R_1 < \rho < R_2$ with center at α.

Remark The sum $\sum_{k=1}^{\infty} c_{-k}(z-\alpha)^{-k}$ is called the singular part (or principal part), and the sum $\sum_{k=0}^{\infty} c_k (z-\alpha)^k$ is called the analytic part of $f(z) = \sum_{k=-\infty}^{\infty} c_k (z-\alpha)^k$ near α. Note that the principal part does not contain a nonzero constant term, and recall that

- α is a removable singularity iff $c_k = 0$ for $k < 0$,
  e.g. $f(z) = \dfrac{\sin z}{z} = \displaystyle\sum_{k=0}^{\infty} \dfrac{(-1)^k z^{2k}}{(2k+1)!}$ has a removable singularity at $z = 0$.
- α is a pole of order n iff $c_k = 0$ for $k < -n$, and $c_{-n} \neq 0$.
- α is an essential singularity iff $c_k \neq 0$ for infinitely many negative k.

Proposition If $f(z)$ has a pole of order k at α, then $\dfrac{1}{f(z)}$ is analytic near α and has a zero of order k at α.

Proof Write

$$f(z) = \frac{g(z)}{(z-\alpha)^k}.$$

Then $g(z)$ is analytic and $g(z) \neq 0$ on a neighborhood of α, so

$$\frac{1}{f(z)} = \frac{(z-\alpha)^k}{g(z)}$$

is analytic on a neighborhood of α and has a zero of order k at α.

Example Let $f(z) = \dfrac{1}{e^z - 1}$. Since $e^0 = 1$, and $\lim_{z\to 0} zf(z) = 1$, f has a simple pole at $z = 0$. Thus $f(z)$ has a Laurent series expansion $\displaystyle\sum_{n=-1}^{\infty} c_n z^n$ about $z = 0$ with $c_1 = 1$. Now, as both

$$g(z) = \frac{z}{e^z - 1} = zf(z) = \sum_{n=-1}^{\infty} c_n z^{n+1} = \sum_{n=0}^{\infty} c_{n-1} z^n \quad \text{and} \quad \frac{1}{g(z)} = \frac{e^z - 1}{z} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$$

are analytic at $z = 0$, we have

$$\left(\sum_{n=0}^{\infty} c_{n-1} z^n\right)\left(\sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}\right) = g(z) \cdot \frac{1}{g(z)} = 1$$

$$\Longrightarrow \left(c_{-1} + c_0 z + c_1 z^2 + \cdots + c_{n-1} z^n + \cdots\right)\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots + \frac{z^n}{(n+1)!} + \cdots\right) = 1$$

$$\Longrightarrow c_{-1} + \left(\frac{c_{-1}}{2!} + c_0\right) z + \left(\frac{c_{-1}}{3!} + \frac{c_0}{2!} + c_1\right) z^2 + \cdots + \left(\frac{c_{-1}}{(n+1)!} + \frac{c_0}{n!} + \cdots + c_{n-1}\right) z^n + \cdots = 1$$

$$\Longrightarrow c_{-1} = 1 \quad \text{and} \quad \frac{c_{-1}}{(n+1)!} + \frac{c_0}{n!} + \sum_{k=2}^{n} \frac{c_{k-1}}{(n-(k-1))!} = \sum_{k=0}^{n} \frac{c_{k-1}}{(n-(k-1))!} = 0 \text{ for } n \geq 1$$

In particular, $c_{-1} = 1$, $c_0 = -1/2$, $c_1 = 1/12$, etc. Finally, in a deleted neighbourhood of $z = 0$, it is straightforward to verify that

$$\sum_{n=1}^{\infty} c_n z^n = f(z) - \frac{c_{-1}}{z} - c_0 = \frac{1}{e^z - 1} - \frac{1}{z} + \frac{1}{2} \quad \text{which converges absolutely on } \mathbb{C} \setminus \{0\}$$

where $\dfrac{1}{e^z - 1} - \dfrac{1}{z} + \dfrac{1}{2}$ is an odd function since

$$\frac{1}{e^{-z} - 1} - \frac{1}{(-z)} + \frac{1}{2} = \frac{e^z}{1 - e^z} + \frac{1}{z} + \frac{1}{2} = -\frac{e^z - 1 + 1}{e^z - 1} + \frac{1}{z} + \frac{1}{2} = -\left(\frac{1}{e^z - 1} - \frac{1}{z} + \frac{1}{2}\right) \quad \forall z \in \mathbb{C} \setminus \{0\}$$

So, $c_{2n} = 0$ for all $n \geq 1$, and the previous recurrence relation can be rewritten as $c_{-1} = 1$, $c_0 = -1/2$ and

$$\frac{1}{(2n+1)!} - \frac{1}{2\,(2n)!} + \sum_{k=1}^{n} \frac{c_{2k-1}}{(2n-(2k-1))!} = 0 \quad \text{for } n \geq 1$$

Remark The coefficients B_n of the Taylor expansion

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n z^n}{n!}$$

are known as Bernoulli numbers. Thus $c_n = B_{n+1}/(n+1)!$ for every $n \geq 1$.

§2 Partial Fractions and Factorization

2.1 Partial Fractions

Theorem Let f be meromorphic in $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, i.e. analytic in $\widehat{\mathbb{C}}$ except at isolated poles. Suppose that $\lim_{z\to\infty} f(z) = \infty$, i.e. f has a pole at ∞. Then $f(z)$ is a rational function, i.e. $f(z) = P(z)/Q(z)$, where P and Q are polynomials.

Proof Since

$$\lim_{z\to\infty} f(z) = \infty \iff \lim_{z\to 0} f(1/z) = \infty,$$

which implies that

- $f(1/z)$ has a pole at $z = 0$,
- there is a disk $D_\varepsilon(0) = \{z \in \mathbb{C} \mid |z| < \varepsilon\}$ such that

$$f(1/z) \text{ has no other pole in } D_\varepsilon(0) \iff f(z) \text{ has no other pole in } \mathbb{C} \setminus D_{1/\varepsilon}(0).$$

Since $\bar{D}_{1/\varepsilon}(0) = \{z \in \mathbb{C} \mid |z| \leq 1/\varepsilon\}$ is compact and all poles are isolated, there are only finitely many poles $z_1, z_2, \ldots, z_n$ of $f(z)$ in $\bar{D}_{1/\varepsilon}(0)$ (and in $\mathbb{C}$). Note that

- at each pole $z_k \in \mathbb{C}$,
$$f(z) = P_k\left(\frac{1}{z - z_k}\right) + G_k(z),$$
where $P_k\left(1/(z - z_k)\right)$ is the principal part of f around z_k and G_k is analytic on a neighborhood $D_{r_k}(z_k)$ of z_k.
- at $z = \infty$,
$$f\left(\frac{1}{z}\right) = P_\infty\left(\frac{1}{z}\right) + G_\infty(z),$$
where as before, $G_\infty(z)$ is analytic in a neighborhood $B_\varepsilon(0)$ of $z = 0$.

and the function

$$H(z) = f(z) - P_\infty(z) - \sum_{k=1}^{n} P_k\left(\frac{1}{z - z_k}\right)$$

is entire and bounded, so, by the Liouville's Theorem, $H(z)$ is a constant and $f(z)$ is a rational function.

Theorem 4. (Mittag-Leffler Theorem) Let $(\zeta_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{C}$ such that $\lim_{k\to\infty} \zeta_k = \infty$ and $P_k(\zeta)$ be polynomials without constant term. Then there exist functions f meromorphic in $\mathbb{C}$ with poles at just the points ζ_k and corresponding singular parts $P_k\left(1/(z-\zeta_k)\right)$.

The most general f of this kind can be written

$$f(z) = g(z) + \sum_k \left[P_k\left(\frac{1}{z-\zeta_k}\right) - p_k(z)\right] \tag{1}$$

where g is analytic and the p_k are polynomials.

Proof Without loss of generality, we assume $\zeta_k \neq 0$ for all k.

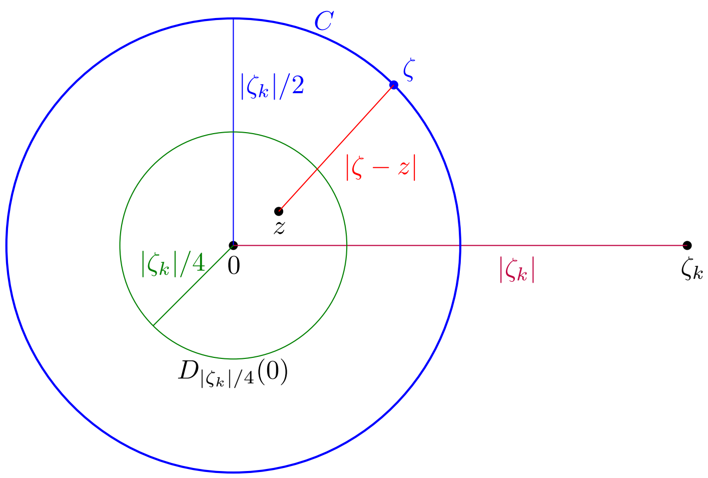
Since $P_k\left(1/(z-\zeta_k)\right)$ is analytic for $|z| < |\zeta_k|$, we can apply the Taylor formula (4.3.1 Theorem 8) to expand $\psi(z) := P_k\left(1/(z-\zeta_k)\right)$ around $z=0$ and write

$$\psi(z) = \sum_{\ell=0}^{N_k} \frac{\psi^{(\ell)}(0)}{\ell!} z^{\ell} + \psi_{N_k+1}(z)\, z^{N_k+1} =: p_k(z) + \psi_{N_k+1}(z)\, z^{N_k+1}$$

for an N_k to be specified shortly, and

$$\psi_{N_k+1}(z) = \frac{1}{2\pi i}\int_C \frac{\psi(\zeta)}{\zeta^{N_k+1}(\zeta - z)}\, d\zeta$$

Let C to be the circle with radius $|\zeta_k|/2$ and center 0, and M_k be the maximum of $|\psi|$ on C.



For each $z \in \overline{D}_{|\zeta_k|/4}(0)$, since $|\zeta - z| \geq |\zeta_k|/4$,

$$|\psi_{N_k+1}(z)| \leq \frac{1}{2\pi}\frac{2\pi|\zeta_k|}{2}\frac{M_k}{(|\zeta_k|/2)^{N_k+1}\,(|\zeta_k|/4)} = 2M_k\left(\frac{2}{|\zeta_k|}\right)^{N_k+1},$$

$$\Longrightarrow |\psi(z) - p_k(z)| := \left|\psi(z) - \sum_{\ell=0}^{N_k}\frac{\psi^{(\ell)}(0)}{\ell!} z^{\ell}\right| = \left|\psi_{N_k+1}(z) z^{N_k+1}\right| \leq 2M_k\left(\frac{2|z|}{|\zeta_k|}\right)^{N_k+1} \leq M_k 2^{-N_k}$$

$$\Longrightarrow \left|P_k\left(\frac{1}{z-\zeta_k}\right) - p_k(z)\right| = |\psi(z) - p_k(z)| \leq 2^{-k} \text{ by choosing } N_k \text{ such that } M_k 2^k \leq 2^{N_k} \tag{2}$$

Claim : $\sum_k \left[P_k\left(\frac{1}{z-\zeta_k}\right) - p_k(z)\right]$ converges uniformly in each disk $\overline{D}_R(0)$ (except at the poles) to a meromorphic function in $\mathbb{C}$.

Proof of Claim For each $R > 0$, we write

$$\sum_k \left[P_k\left(\frac{1}{z-\zeta_k}\right) - p_k(z)\right] = \sum_{|\zeta_k|/4 \le R} \left[P_k\left(\frac{1}{z-\zeta_k}\right) - p_k(z)\right] + \sum_{|\zeta_k|/4 > R} \left[P_k\left(\frac{1}{z-\zeta_k}\right) - p_k(z)\right]$$

Since $\overline{D}_R(0)$ is compact,

- the first term on the right-hand side corresponds to a finite sum and has $P_k\left(1/(z-\zeta_k)\right)$ as the singular part at the pole ζ_k,
- the second term on the right-hand side converges uniformly to an analytic function in $\overline{D}_R(0)$ by Eq.(2) and the Weierstrass' Theorem,

the function h defined by

$$h(z) := \sum_k \left[P_k\left(\frac{1}{z-\zeta_k}\right) - p_k(z)\right] \quad \text{is a meromorphic function in } \mathbb{C}.$$

Finally, if f is a meromorphic function with the same poles ζ_k and same singular parts as h, $g = f - h$ is analytic and $f = g + h$. $\square$

2.2 Infinite Products

Definition An infinite product $\prod_{n=1}^{\infty} a_n$ of complex numbers *converges if* there exists $N \ge 1$ such that $a_k \neq 0$ for all $k \ge N$ and the limit of partial products

$$a = \lim_{n\to\infty} \prod_{k=N}^{n} a_k$$

exists and is nonzero. In this case,

$$\prod_{n=1}^{\infty} a_n := \left(\prod_{n=1}^{N-1} a_n\right) a$$

Examples

i. $\prod_{k=1}^{\infty}(1+1/k) = \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\ldots$ diverges (to ∞) since $\lim_{n\to\infty}\prod_{k=1}^{n}(1+1/k) = \lim_{n\to\infty} n = \infty$.

ii. $\prod_{k=2}^{\infty}(1-1/k) = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\ldots$ diverges to 0,

iii. $\prod_{k=2}^{\infty}\left(1-1/k^2\right) = \prod_{k=2}^{\infty}(k-1)(k+1)/k^2$ converges (to $1/2$), $\prod_{k=1}^{\infty}\left(1-1/k^2\right)$ converges (to 0).

Remark Note that

- $\prod_{n=1}^{\infty} a_n$ converges iff at most a finite number of $a_n's$ are zero, and if the partial products formed by the nonvanishing $a_k's$ tend to a finite nonzero limit a.
- if $\prod_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 1 \iff$ if $\prod_{n=1}^{\infty}(1+a_n)$ converges, then $\lim_{n\to\infty} a_n = 0$.

Theorem 5 Let $1+a_n \neq 0$ for all $n \in \mathbb{N}$. Then $\prod_{n=1}^{\infty}(1+a_n)$ converges iff $\sum_{n=1}^{\infty} \mathrm{Ln}\,(1+a_n)$ converges, where Ln is the principal branch of the logarithm, i.e. $-\pi < \mathrm{Im}(\mathrm{Ln}\, z) = \mathrm{Arg} z \leq \pi$.

Remark That is the question of the convergence of infinite products can be reduced to the question of the convergence of infinite sums.

Proof $(\Longleftarrow)$ Let

$$S_n = \sum_{k=1}^{n} \mathrm{Ln}\,(1+a_k) = \sum_{k=1}^{n} [\log|1+a_k| + i\mathrm{Arg}\,(1+a_k)] \quad \text{and} \quad P_n := \prod_{k=1}^{n}(1+a_k) = e^{S_n}$$

Hence, by the continuity of e^z,

$$\text{if} \quad \sum_{n=1}^{\infty} \mathrm{Ln}\,(1+a_n) = \lim_{n\to\infty} S_n = S \implies \prod_{n=1}^{\infty}(1+a_n) = \lim_{n\to\infty} P_n = \lim_{n\to\infty} e^{S_n} = e^S = P \neq 0$$

$(\Longrightarrow)$ If $\lim_{n\to\infty} P_n = P \neq 0$, since $P_n = e^{S_n} = e^{S_n + 2m\pi i}$ $\forall\, m \in \mathbb{Z}$, there exists $M_n \in \mathbb{Z}$ for each $n \in \mathbb{N}$ such that

$$-\pi < \sum_{k=1}^{n} \mathrm{Arg}(1+a_k) - \mathrm{Arg}\, P + 2\pi M_n \leq \pi \quad \text{for all } n \in \mathbb{N},$$

$$\text{and} \quad \mathrm{Ln}\left(\frac{P_n}{P}\right) = S_n - \mathrm{Ln}P + 2\pi i\, M_n, \quad \mathrm{Ln}\left(\frac{P_{n+1}}{P}\right) = S_{n+1} - \mathrm{Ln}P + 2\pi i\, M_{n+1}$$

$$\implies 2\pi i\ (M_{n+1} - M_n) = \mathrm{Ln}\left(\frac{P_{n+1}}{P}\right) - \mathrm{Ln}\left(\frac{P_n}{P}\right) - \mathrm{Ln}\,(1+a_{n+1})$$

$$\implies \begin{cases} 0 = \mathrm{Ln}\left|\frac{P_{n+1}}{P}\right| - \mathrm{Ln}\left|\frac{P_n}{P}\right| - \mathrm{Ln}\,|1+a_{n+1}| \\ 2\pi\ (M_{n+1} - M_n) = \mathrm{Arg}\left(\frac{P_{n+1}}{P}\right) - \mathrm{Arg}\left(\frac{P_n}{P}\right) - \mathrm{Arg}\,(1+a_{n+1}) \end{cases}$$

Now,

$$\lim_{n\to\infty}\left[\mathrm{Arg}\left(\frac{P_{n+1}}{P}\right) - \mathrm{Arg}\left(\frac{P_n}{P}\right)\right] = 0 \quad \text{since} \ \lim_{n\to\infty}\frac{P_n}{P} = 1$$

$$\implies \lim_{n\to\infty} 2\pi\ |M_{n+1} - M_n| = \lim_{n\to\infty} |\mathrm{Arg}\,(1+a_{n+1})|$$

$$\implies \lim_{n\to\infty} 2\pi\ |M_{n+1} - M_n| = 0 \quad \text{since} \ |\mathrm{Arg}\,(1+a_{n+1})| \leq \pi, \ \text{and} \ M_n,\ M_{n+1} \in \mathbb{Z}$$

We must have $M_{n+1} = M_n$ for n large enough, i.e. for n sufficiently large, $M_n = M \in \mathbb{Z}$. So, for such large n, since $\lim_{n\to\infty} P_n/P = 1$,

$$\mathrm{Ln}\left(\frac{P_n}{P}\right) = S_n - \mathrm{Ln}P + 2\pi i\, M \implies \sum_{n=1}^{\infty} \mathrm{Ln}\,(1+a_n) = \lim_{n\to\infty} S_n = \mathrm{Ln}P - 2\pi i M \qquad \square$$

Definition The infinite product $\prod_{n=1}^{\infty}(1+a_n)$ is said to be *absolutely convergent* if the infinite sum $\sum_{n=1}^{\infty} \mathrm{Ln}\,(1+a_n)$ is absolutely convergent.

Theorem 6 The product $\prod_{n=1}^{\infty}(1+a_n)$ is absolutely convergent iff $\sum_{n=1}^{\infty}|a_n|$ converges, that is $\sum_{n=1}^{\infty}\operatorname{Ln}(1+a_n)$ converges absolutely iff $\sum_{n=1}^{\infty}|a_n|$ converges.

Remark Since $|-a_n|=|a_n|$, $\prod_{n=1}^{\infty}(1+a_n)$ is called *absolutely convergent* if $\prod_{n=1}^{\infty}(1+|a_n|)$ converges.

Proof Since the convergence for $\sum_{n=1}^{\infty}\operatorname{Ln}(1+a_n)$ or $\sum_{n=1}^{\infty}|a_n|$ implies $\lim_{n\to\infty}a_n=0$, and since

$$\lim_{z\to 0}\frac{\operatorname{Ln}(1+z)}{z}=1 \implies \exists\, N\in\mathbb{N} \text{ such that, if } n\geq N, \text{ then } \frac{1}{2}|a_n|\leq|\operatorname{Ln}(1+a_n)|\leq\frac{3}{2}|a_n|$$

Thus, we have $$\sum_{n=1}^{\infty}|a_n| \text{ converges} \iff \sum_{n=1}^{\infty}|\operatorname{Ln}(1+a_n)| \text{ converges}$$

Remark We wish to consider analytic functions defined by infinite products; i.e., functions of the form

$$f(z)=\prod_{k=1}^{\infty}(1+u_k(z))$$

Recall that f is analytic if each function u_k, $k=1,2,\ldots$ is analytic and the partial products converge to their limit function uniformly on compacta.

Theorem Suppose that $u_k(z)$ is analytic in an open connected subset Ω for $k=1,2,\ldots,$ and that $\sum_{k=1}^{\infty}|u_k(z)|$ converges uniformly on compacta. Then the product $\prod_{k=1}^{\infty}(1+u_k(z))$ converges uniformly on compacta and represents an analytic function in Ω.

Proof Let D be a compact subset of Ω.

Since $\sum_{k=1}^{\infty}|u_k(z)|$ converges uniformly on D,

- for sufficiently large k, $|u_k(z)|<1$ there. Hence, we may assume that $1+u_k\neq 0$ for all k.
- If we then take N large enough so that $\sum_{k=N+1}^{\infty}|u_k(z)|<\varepsilon/2$, it follows, as in the proof of

Theorem 6, that

$$\left|\sum_{k=N+1}^{\infty}\operatorname{Ln}(1+u_k(z))\right|\leq\varepsilon \quad \text{throughout } D.$$

i.e., $\sum_{k=1}^{\infty}\operatorname{Ln}(1+u_k(z))$ converges uniformly on D to a limit function $S(z)$. It follows that $S(D)$ is bounded. Finally, since the exponential function is uniformly continuous in any bounded domain,

$$P_N(z)=\exp\left(\sum_{k=1}^{N}\operatorname{Ln}(1+u_k(z))\right)=\prod_{k=1}^{N}(1+u_k(z))$$

converges uniformly to its limit function $e^{S(z)}$. $\square$

Examples

1. $\displaystyle\sum_{k=1}^{\infty}\left(1+z^k\right)$ converges uniformly on any compact subset of the unit disk $D_1(0)$ since any compact subset is contained in a disk $D_\delta(0)$ of radius $\delta < 1$. Hence

$$\sum_{k=1}^{\infty}|z^k| \le \sum_{k=1}^{\infty}\delta^k = \frac{\delta}{1-\delta}$$

and, by the M-test, $\displaystyle\sum_{k=1}^{\infty}|z^k|$ is uniformly convergent.

2. $\displaystyle\prod_{k=1}^{\infty}(1+1/k^z)$ represents an analytic function in the half-plane $H=\{z\in\mathbb{C}\mid \operatorname{Re} z>1\}$. In any compact subset of H, $\operatorname{Re} z \ge 1+\delta$ throughout so that

$$\left|\frac{1}{k^z}\right| = \frac{1}{k^{\operatorname{Re} z}} \le \frac{1}{k^{1+\delta}}, \quad k=1,\, 2, \ldots$$

Hence $\displaystyle\sum_{k=1}^{\infty}\left|\frac{1}{k^z}\right|$ and, consequently, $\displaystyle\prod_{k=1}^{\infty}\left(1+\frac{1}{k^2}\right)$ are uniformly convergent.

2.3 Canonical Products

If g is an entire function, $f(z) := e^{g(z)}$ is entire and everywhere nonzero. Conversely, if f is an entire, nonzero function, then, since f'/f is an entire function which has a primitive g that is also entire such that $g' = f'/f$ and

$$\frac{d}{dz}\left[f(z)e^{-g(z)}\right] = f'(z)e^{-g(z)} - f(z)\frac{f'(z)}{f(z)}e^{-g(z)} = 0 \implies f(z)e^{-g(z)} = C \in \mathbb{C}$$

so, $f(z)$ is of the form $f(z) = e^{g(z)}$ for some entire function $g(z)$.

By this method we can also find the most general entire function with a finite number of zeros. Assume that $f(z)$ has M zeros at the origin (M may be zero), and denote the other zeros by $a_1,\, a_2, \ldots,\, a_N$, multiple zeros being repeated. Then

$$f(z) = z^M e^{g(z)}\prod_{n=1}^{N}\left(1-\frac{z}{a_n}\right) \quad \text{for some entire function } g.$$

If there are infinitely many zeros, we can try to obtain a similar representation by means of an infinite product. the obvious generalization would be

$$f(z) = z^M e^{g(z)}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right). \tag{3}$$

Indeed, if $\displaystyle\sum_{n=1}^{\infty}1/|a_n|$ converges, $\displaystyle\sum_{n=1}^{\infty}|z/a_n|$ converges uniformly on every compact set, so that the product $\displaystyle\prod_{n=1}^{\infty}(1-z/a_n)$ is uniform convergent on compacta and gives the desired entire function.

Moreover, if $\sum_{n=1}^{\infty} 1/|a_n|$ diverges, but $\sum_{n=1}^{\infty} 1/|a_n|^2$ converges, we can modify the above construction by considering

$$f(z) = z^M e^{g(z)} \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{a_n}\right) e^{z/a_n} \right]. \tag{4}$$

With the "convergence factors" e^{z/a_n}, the product is uniformly convergent on compacta since, for $|a_n| > 2|z| \iff |z|/|a_n| < 1/2$,

$$\begin{aligned}\left|\log\left[\left(1 - \frac{z}{a_n}\right) e^{z/a_n}\right]\right| &= \left|\left(-\frac{z}{a_n} - \frac{1}{2}\left(\frac{z}{a_n}\right)^2 - \frac{1}{3}\left(\frac{z}{a_n}\right)^3 - \cdots\right) + \frac{z}{a_n}\right| \\ &\le \left|\frac{z^2}{a_n^2}\right| \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) = \left|\frac{z^2}{a_n^2}\right|.\end{aligned}$$

Hence the series

$$\sum_{n=1}^{\infty} \log\left[(1 - z/a_n)\, e^{z/a_n}\right], \quad z \neq a_n$$

is uniformly convergent and the product is uniformly convergent on compacta.

By the same reasoning, if $\sum_{n=1}^{\infty} 1/|a_n|^{N_n+1}$ converges for some integer N_n, then the infinite product

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \frac{1}{3}\left(\frac{z}{a_n}\right)^3 + \cdots + \frac{1}{N_n}\left(\frac{z}{a_n}\right)^{N_n}} \tag{5}$$

is uniformly convergent on compacta and represents an entire function with the desired zeros. However, since there are sequences $(a_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} a_n = \infty$ and yet $\sum_{n=1}^{\infty} 1/|a_n|^N$ diverges for all N, e.g. $(a_n)_{n=2}^{\infty} = (\log n)_{n=2}^{\infty}$, we must introduce a slight variation for the general case.

Theorem 7 (Weierstrass Factorization Theorem) There exists an entire function with arbitrarily prescribed zeros $(a_n)_{n=1}^{\infty}$, as long as $\lim_{n\to\infty} a_n = \infty$ if the numbers of zeros is infinite. Moreover, every entire function with these and no other zeros can be written as

$$f(z) = z^M e^{g(z)} \prod_{n=1}^{\infty} (1 - z/a_n)\, e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{N_n}\left(\frac{z}{a_n}\right)^{N_n}} \tag{6}$$

where the product is taken over all $a_n \neq 0$, the N_n are integers, and g is an entire function.

Proof Let $(a_n)_{n=1}^{\infty}$ be a sequence of complex numbers such that $a_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \infty$. We shall prove the existence of polynomials $p_n(z)$ such that

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{p_n(z)} \tag{7}$$

converges to an entire function. For each $n \in \mathbb{N}$, let

$$r_n(z) := \log\left[\left(1 - \frac{z}{a_n}\right) e^{p_n(z)}\right] = \log\left(1 - \frac{z}{a_n}\right) + p_n(z)$$

where the branch of the logarithm shall be chosen so that $-\pi < \operatorname{Im} r_n(z) \le \pi$.

For a given R we consider only the terms with $|a_n| > R \iff (R/|a_n|) < 1$. In the disk $|z| \le R$ the principal branch of $\log(1 - z/a_n)$ can be developed in a Taylor series

$$\log\left(1 - \frac{z}{a_n}\right) = -\frac{z}{a_n} - \frac{1}{2}\left(\frac{z}{a_n}\right)^2 - \frac{1}{3}\left(\frac{z}{a_n}\right)^3 - \cdots$$

We reverse the signs and choose $p_n(z)$ as a partial sum

$$p_n(z) = \frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \frac{1}{3}\left(\frac{z}{a_n}\right)^3 + \cdots + \frac{1}{N_n}\left(\frac{z}{a_n}\right)^{N_n}$$

Then $r_n(z)$ has the representation

$$r_n(z) = -\frac{1}{N_n+1}\left(\frac{z}{a_n}\right)^{N_n+1} - \frac{1}{N_n+2}\left(\frac{z}{a_n}\right)^{N_n+2} - \cdots$$

and we obtain easily the estimate

$$|r_n(z)| \le \frac{1}{N_n+1}\left(R/|a_n|\right)^{N_n+1}\frac{1}{1 - R/|a_n|} \tag{8}$$

Suppose now the series

$$\sum_{n=1}^{\infty}\frac{1}{N_n+1}\left(R/|a_n|\right)^{N_n+1} \tag{9}$$

converges. By the estimate (8) it follows that $\lim_{n\to\infty} r_n(z) = 0$, and hence $-\pi < \operatorname{Im} r_n(z) \le \pi$ as soon as n is sufficiently large. Moreover, the comparison shows that the series $\sum r_n(z)$ is absolutely and uniformly convergent for $|z| \le R$, and thus the product (7) represents an analytic function in $|z| < R$. For the sake of reasoning we had to exclude the values $|a_n| \le R$, but it is clear that the uniform convergence of (7) is not affected when the corresponding factors are again taken into account.

It remains only to show that the series (9) can be made convergent for all R. But this is obvious, for if we take $N_n = n$ it is clear that (9) has a majorant geometric series with ratio < 1 for any fixed value of R. $\square$

Corollary Every function which is meromorphic on all of $\mathbb{C}$ is the quotient of two entire functions.

Indeed, if f is meromorphic on $\mathbb{C}$, the theorem enables us to construct an entire function g whose zeros are the poles of f. $F(z) := f(z)g(z)$ is then an entire function, and

$$f(z) = \frac{F(z)}{g(z)}$$

Example 1. To find an entire function f with a single zero at every negative integer $a_k = -k$, note that $\sum_{k=1}^{\infty} 1/|a_k|$ diverges but $\sum_{k=1}^{\infty} 1/|a_k|^2$ converges so that we can define $f(z) = \prod_{k=1}^{\infty}(1 + z/k)\, e^{-z/k}$.

Example 2. An entire function with zeroes at all the points $a_k = \log k$, $k = 1, 2, \ldots$, is given by

$$f(z) = z\prod_{k=2}^{\infty}\left(1 - \frac{z}{\log\, k}\right)\exp\left(\frac{z}{\log\, k} + \frac{z^2}{2\log^2\, k} + \cdots + \frac{z^k}{k\log^k\, k}\right).$$

Example 3. An entire function with a single zero at every integer is given by

$$f(z) = z\prod_{k=1}^{\infty}\left[(1-z/k)\, e^{z/k}\,(1+z/k)\, e^{-z/k}\right] = z\prod_{k=1}^{\infty}\left(1-z^2/k^2\right).$$

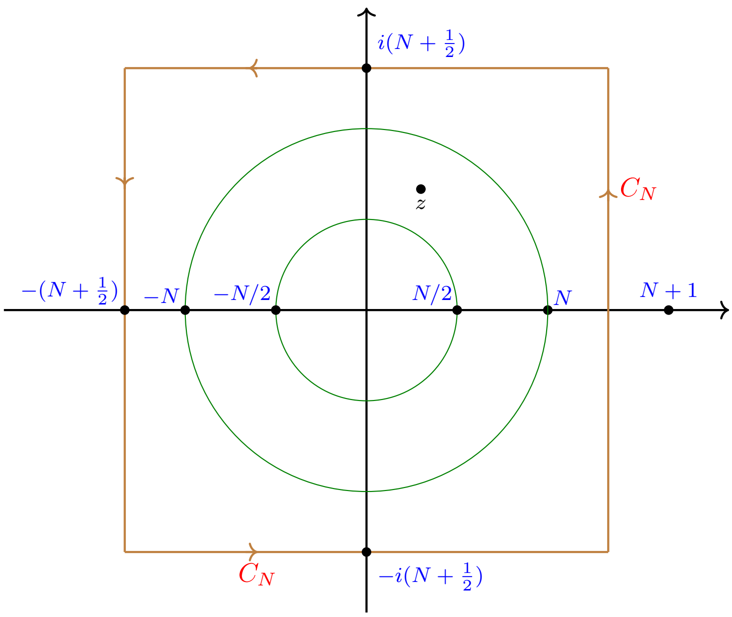
Proposition Let $f(z) = z\prod_{k=1}^{\infty}\left(1-z^2/k^2\right)$. Then $f(z) = (\sin \pi z)/\pi$.

Proof Consider the quotient $Q(z) = z\prod_{k=1}^{\infty}\left(1-z^2/k^2\right)/\sin \pi z$. Note that

- Q is entire and zero-free.
- to show that Q is constant we seek estimates on its growth for large z, and assume then that $N/2 \le |z| \le N$.

By the Maximum Modulus Theorem, $|Q(z)|$ is bounded by the maximum value assumed by Q on the square of side $2N+1$ centered at the origin. We have already proved that along this square (which avoids the zeros of $\sin \pi z$), $|1/\sin \pi z| \le 4$. Moreover, since $e^x \ge 1+x$ for all $x \in \mathbb{R}$,

$$\begin{aligned}\left|\prod_{k=1}^{\infty}\left(1-z^2/k^2\right)\right| &= \left|\prod_{k=1}^{N}(1-z/k)(1+z/k)\prod_{k=N+1}^{\infty}\left(1-z^2/k^2\right)\right| \le \prod_{k=1}^{N} e^{2|z/k|}\prod_{k=N+1}^{\infty} e^{|z^2/k^2|}\\ &\le \exp\left(2|z|\,(1+\log N) + |z^2|/N\right)\\ \text{since}\quad &\sum_{k=1}^{N} 1/k = 1 + \sum_{k=2}^{N} 1/k < 1+\log N \text{ and } \sum_{k=N+1}^{\infty} 1/k^2 < \int_N^{\infty}\left(1/x^2\right)dx = 1/N\end{aligned}$$



Also note that for large N, $2\,(1+\log N) < \sqrt{N/2} \le |z|^{1/2}$ while $|z^2|/N \le |z|$, it follows that

$$|Q(z)| = \frac{\left|z\prod_{k=1}^{\infty}\left(1-z^2/k^2\right)\right|}{|\sin \pi z|} \le A\exp\left(|z^{3/2}|\right)$$

By the extended Liouville's Theorem, we must have

$$\frac{z\prod_{k=1}^{\infty}\left(1-z^2/k^2\right)}{\sin\pi z}=A\,e^{Bz}$$

However, Q is an even function so that $B=0$, and the constant A can be determined by noting that $A=Q(0)=\lim_{z\to 0}(z/\sin\pi z)=1/\pi$. $\square$

Some consequences of the above proposition:

i. Setting $z=1/2$, we have

$$\begin{aligned}1&=\frac{\pi}{2}\prod_{k=1}^{\infty}\left[1-1/\left(2k\right)^2\right]\\ \iff \frac{2}{\pi}&=\left(\frac{1\cdot 3}{2\cdot 2}\right)\left(\frac{3\cdot 5}{4\cdot 4}\right)\left(\frac{5\cdot 7}{6\cdot 6}\right)\cdots \iff \pi=2\cdot\left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right)\cdots\end{aligned}$$

ii. Suppose we expand the terms in the product to obtain an infinite series. Then we will have

$$\begin{aligned}\sin\pi z&=\pi z\prod_{k=1}^{\infty}\left(1-z^2/k^2\right)\\ &=\pi z\left[1-\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)z^2+2\left(\sum_{k,\,j=1}^{\infty}\frac{1}{k^2j^2}\right)z^4-+\cdots\right]\end{aligned}$$

A comparison with the familiar (Maclaurin or Taylor) power series

$$\sin\pi z=\pi z-\frac{\pi^3z^3}{6}+\frac{\pi^5z^5}{120}-+\cdots$$

shows that $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$.

Genus of a Canonical Product

The proof of the Weierstrass Factorization Theorem has shown that the *canonical product*

$$\prod_{n=1}^{\infty}\left(1-z/a_n\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{h}\left(\frac{z}{a_n}\right)^h}\quad\text{where } h \text{ is independent of } n \tag{10}$$

converges and represents an entire function provided that the series $\displaystyle\sum_{n=1}^{\infty}\left(R/|a_n|\right)^{h+1}/\left(h+1\right)$ converges for all R, that is to say provided that $\displaystyle\sum_{n=1}^{\infty}1/|a_n|^{h+1}$ converges.

Definition The expression (10) is called a *canonical product* associated with the sequence $(a_n)_{n=1}^{\infty}$, and h is called the *genus* of the canonical product if h is the smallest integer such that $\displaystyle\sum_{n=1}^{\infty}1/|a_n|^{h+1}$ converges.

Definition Let $f(z)$ be an entire function with arbitrarily prescribed zeros

$$(a_n)_{n=1}^{\infty} \quad , \quad \lim_{n\to\infty} a_n = \infty.$$

If $f(z)$ can be represented in the form

$$f(z) = z^M e^{g(z)} \prod_{n=1}^{\infty} (1 - z/a_n)\, e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{h}\left(\frac{z}{a_n}\right)^h} \tag{11}$$

where $g(z)$ is a polynomial of degree $\deg g$, and the product is taken over all $a_n \neq 0$, then $f(z)$ is said to be of *finite genus* and the *genus of $f(z)$* is defined to be $\max\{\deg g,\, h\}$, the maximum between the degree of $g(z)$ and the genus of the canonical product. If there is no such representation, the genus of $f(z)$ is infinite.

Examples

• The canonical representation of a genus $0 = \max\{\deg g, h\}$ entire function is of the form

$$f(z) = Az^M \prod_{n=1}^{\infty} (1 - z/a_n) \quad \text{for some } A \in \mathbb{C}$$

• The canonical representation of genus $1 = \max\{\deg g, h\}$ entire functions can have either of the following two forms:

$$(1) \quad f(z) = Bz^M e^{\alpha z} \prod_{n=1}^{\infty} (1 - z/a_n)\, e^{z/a_n} \quad \text{for some } B \in \mathbb{C}^* \;,\; \alpha \in \mathbb{C}$$

with $\sum_{n=1}^{\infty} 1/|a_n|$ divergent and $\sum_{n=1}^{\infty} 1/|a_n|^2$ convergent $(h = 1)$.

$$(2) \quad f(z) = Cz^M e^{\alpha z} \prod_{n=1}^{\infty} (1 - z/a_n) \quad \text{for some } C \in \mathbb{C}^* \;,\; \alpha \in \mathbb{C}^*$$

with $\sum_{n=1}^{\infty} 1/|a_n|$ convergent $(h = 0)$.

• Since $\sin \pi z = 0$ for $z = \pm n$, $n \in \mathbb{N} \cup \{0\}$, $\sum_{n=1}^{\infty} 1/n = \infty$ and $\sum_{n=1}^{\infty} 1/n^2 < \infty$ $(h = 1)$, we have

$$\sin \pi z = ze^{g(z)} \prod_{n=1}^{\infty} (1 - z/n)\, e^{z/n} (1 + z/n)\, e^{-z/n} = ze^{g(z)} \prod_{n\neq 0} (1 - z/n)$$

$$\Longrightarrow \pi \cot \pi z = \frac{1}{z} + g'(z) + \sum_{n\neq 0} \frac{1}{z-n}$$

$$\Longrightarrow g'(z) = \pi \cot \pi z - \frac{1}{z} - \sum_{n\neq 0} \frac{1}{z-n} \text{ is bounded and entire}$$

$$\Longrightarrow g'(z) = 0 \Longrightarrow e^{g(z)} = \pi \text{ since } \lim_{z\to 0} \frac{\sin \pi z}{z} = \pi \text{ and } \sin \pi z = \pi z \prod_{n\neq 0} (1 - z/n)$$

Thus $\sin \pi z$ is an entire function of genus 1.

§ 3 Entire Functions

3.2 Hadamard's Theorem

Definition Let f be an entire function, and $M(R) = \max_{|z|=R} |f(z)|$, the maximum of $|f(z)|$ on $|z| = R$. The *order λ of f* is defined by

$$\lambda = \limsup_{n\to\infty} \frac{\log \log M(R)}{\log R} \tag{12}$$

In other words, λ is the smallest number such that

$$M(R) \leq e^{R^{\lambda+\varepsilon}}$$

for all $\varepsilon > 0$, as soon as R is large enough.

Theorem 8 (Hadamard Factorization Theorem) The genus h and the order λ of an entire function satisfy the double inequality $h \leq \lambda \leq h + 1$.

The proof of this theorem is somewhat lengthy, and we will skip it in this course. You can however find the key steps of the proof in Ahlfors' textbook.

Corollary An entire function of fractional order assumes every finite value infinitely many times.

Proof $\forall\, z_0 \in \mathbb{C}$, f and $f - z_0$ have the same order. Thus, to prove the corollary we just want to show than an entire function with fractional order has infinitely many zeros.

Let us assume that f has a finite number of zeros. Then there exists a polynomial p such that $F = \dfrac{f}{p}$ does not have any zeros, and has the same order λ as f. Hadamard's factorization theorem then tells us that $F = e^{g(z)}$, where g is a polynomial of degree h such that $h \leq \lambda \leq h+1$. Now, it is clear that the order of $F = e^{g(z)}$ is h itself, which is an integer. We have a contradiction, which proves the corollary. $\square$

Analytic Functions Defined by Definite Integrals

Recall that Morera's Theorem (in 4.2.3) says that : if f is continuous in an open connected set Ω, and satisfies that $\displaystyle\int_\gamma f(z)\, dz = 0$ for all closed curves γ in Ω, then f is analytic in Ω. So, Morera's Theorem can be used to prove the analyticity of certain functions given in integral form as follows.

Theorem Suppose $\varphi(z,t)$ is a continuous function of t, $a \leq t \leq b$, for fixed z and an analytic function of for each z in the open connected set D for fixed t. Then

$$f(z) = \int_a^b \varphi(z,t)\, dt \quad \text{is analytic in } D \tag{13}$$

Proof Since f is a continuous function of z, according to Morera's Theorem, we need only prove that $\displaystyle\int_\gamma f(z)\, dz = 0$ for any rectangle $\gamma \subset D$. Since φ is continuous in t and analytic in z,

$$\int_\gamma f(z)\, dz = \int_\gamma \left(\int_a^b \varphi(z,t)\, dt \right) dz = \int_a^b \left(\int_\gamma \varphi(z,t)\, dz \right) dt = 0$$

Examples

1. $f(z) = \displaystyle\int_0^1 dt/(t-z)$ is analytic in $D = \mathbb{C} \setminus [0,1]$.
2. $g(z) = \displaystyle\int_0^\infty dt/\left(e^t - z\right)$ is analytic in $D = \mathbb{C} \setminus [1,\infty)$. Although g is given by an improper integral, it is the uniform limit of

$$g_n(z) = \int_0^n \frac{dt}{e^t - z}$$

on any compact subset of $D = \mathbb{C} \setminus [1,\infty)$, and hence g is analytic.

Analytic Functions Defined by Dirichlet Series

Series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z} \quad \text{where } a_n \in \mathbb{C},\ n \in \mathbb{N}$$

are known as *Dirichlet Series.*

Note that $n^{-z} = \exp(-z \log n)$ represents an entire function for every $n \in \mathbb{N}$. ($\log n$ is chosen as the principal value; i.e., $\log n$ is real-valued, so $n^{-z} > 0$ is positive for all $z \in \mathbb{R}$.) Since the partial sums are entire, a function $f(z)$, defined by a Dirichlet series, *is analytic in any region where the series converges uniformly.*

According to the theorems below, the natural regions of convergence for Dirichlet series are half-planes of the form $Re\, z > x_0$, much as disks centered at the origin are the natural regions associated with power series.

Theorem If $\displaystyle\sum_{n=1}^{\infty} a_n/n^z$ converges for $z = z_0$, then it converges for all $z \in H = \{z \mid Re\, z > Re\, z_0\}$. Moreover, the convergence is uniform in any compact subset of the half-plane H.

Proof For each $k \in \mathbb{N}$, $z \in H$, let $A_k = \displaystyle\sum_{n=1}^{k} a_n/n^{z_0}$, $w = z - z_0$ and $b_k = 1/k^w$. Since

- $\displaystyle\sum_{n=1}^{\infty} a_n/n^{z_0}$ converges, there exists an $A > 0$ such that $|A_k| = \left|\displaystyle\sum_{n=1}^{k} a_n/n^{z_0}\right| < A$ for all $k \in \mathbb{N}$,
- $Re\, w = Re\,(z - z_0) > 0$ for $z \in H$, $\displaystyle\lim_{k\to\infty} |b_k| = \lim_{k\to\infty} 1/k^{Re\, w} = 0$ and

$$|b_k - b_{k+1}| = \left|\frac{1}{k^w} - \frac{1}{(k+1)^w}\right| = \left|\int_k^{k+1} w\, t^{-w-1}\, dt\right| < \int_k^{k+1} \frac{|w|\,|dt|}{k^{Re\, w+1}} = \frac{1}{k^{Re\, w+1}} \to 0 \text{ as } k \to \infty,$$

- $$\left|\sum_{k=M}^{N} \frac{a_k}{k^z}\right| = \left|\frac{a_M}{M^z} + \cdots + \frac{a_N}{N^z}\right| = |(A_M - A_{M-1})\, b_M + \cdots + (A_N - A_{N-1})\, b_N| \quad \forall\, M \le N \in \mathbb{N}$$

$$\le |-A_{M-1} b_M| + \sum_{k=M}^{N-1} |A_k|\, |b_k - b_{k+1}| + |A_N b_N|$$

$$\le A\left(|b_M| + \sum_{k=M}^{N-1} |b_k - b_{k+1}| + |b_N|\right) \to 0 \text{ as } M \to \infty$$

the Dirichlet series $\sum_{n=1}^{\infty} a_n/n^z$ converges for all $z \in H$.

Finally, note that if K is a compact subset of H, there is a $\delta > 0$ with $Re(z - z_0) > \delta$ for all $z \in K$, as well as a positive constant B with $|z| < B$ throughout K. Hence the partial sums $\sum_{k=M}^{N} a_k/k^z$ will have a uniformly small absolute value for all $z \in K$, once M is sufficiently large. So the series converges to its limit function uniformly in K. $\square$

Remark Note that in the above proof, we never actually used the convergence of the Dirichlet series at z_0. The only actual requirement for the conclusion was that there was a finite upper bound for the absolute value of its partial sums.

Example Suppose $a_n = (-1)^n$. Then $\sum_{n=1}^{\infty} a_n/n^z$ has bounded partial sums (although it diverges) at $z = 0$. According to the above Theorem, then, it converges and represents an analytic function in the right half-plane: $Re\, z > 0$. The fact that it diverges at $z = 0$ also implies that its partial sums are not bounded for any value of z with a negative real part.

Theorem If $\sum_{n=1}^{\infty} a_n/n^z$ converges for some, but not all, values of z, there exists a real constant x_0 (called the *abscissa of convergence*) such that $\sum_{n=1}^{\infty} a_n/n^z$ converges if $Re\, z > x_0$ and diverges if $Re\, z < x_0$.

Proof Let $x_0 = \inf\{Re\, z \mid \sum_{n=1}^{\infty} a_n/n^z \text{ converges}\}$. By the above Theorem, if $x_0 = \infty$, the series converges for all z. If the series neither converges for all z nor diverges for all z, $-\infty < x_0 < \infty$ and the theorem follows from the above Theorem.

Theorem Suppose $\sum_{n=1}^{\infty} a_n/n^z$ converges absolutely for some, but not all, values of z, there exists a real constant x_1 (called the *abscissa of absolute convergence*) such that $\sum_{n=1}^{\infty} a_n/n^z$ converges absolutely if $Re\, z > x_1$ and does not converge absolutely if $Re\, z < x_1$.

Example The function $\zeta(z)$ is defined by the Dirichlet series $\sum_{n=1}^{\infty} 1/n^z$. This series converges absolutely for $Re\, z > 1$, and diverges if $Re\, z < 1$.

Since Dirichlet series converge uniformly within their half-plane of convergence, they can be differentiated term-by-term. So if $f(z) = \sum_{n=1}^{\infty} a_n/n^z$, then

$$f'(z) = \sum_{n=1}^{\infty} \frac{-a_n \log n}{n^z}.$$

For any value of z within the half-planes of convergence for two Dirichlet series, we have :

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z} + \sum_{n=1}^{\infty} \frac{b_n}{n^z} = \sum_{n=1}^{\infty} \frac{a_n + b_n}{n^z}.$$

We can also multiply two Dirichlet series. Rewriting the product as another Dirichlet series involves a rearrangement of the terms, which is justified if the two series are absolutely convergent. Hence, within the half-planes of *absolute* convergence, we have

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z} \cdot \sum_{n=1}^{\infty} \frac{b_n}{n^z} = \sum_{n=1}^{\infty} \frac{c_n}{n^z} \quad \text{where} \quad c_n = \sum_{d\,|\,n} a_d b_{n/d}$$

i.e. c_n is defined as the "convolution" of a_n and b_n, and the sum is taken over all the positive divisors of n.

Example

$$\zeta^2(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} \frac{d(n)}{n^z}$$

where $d(n)$ equals the number of positive divisors of n.

2.4 The Gamma Function

Lemma If $s_n = 1 + 1/2 + \cdots + 1/n - \log n$, then $\lim\limits_{n\to\infty} s_n$ exists. This limit is called the *Euler constant*, γ.

Proof $t_n = 1 + 1/2 + \cdots + 1/n - \log n$increases with n. Geometrically this is obvious since t_n represents the area of the $n-1$ regions between the upper Riemann sum and the exact value for $\displaystyle\int_1^n (1/x)dx$. We can write

$$t_n = \sum_{k=1}^{n-1} \left[\frac{1}{k} - \log\left(\frac{k+1}{k}\right)\right] \quad \text{and} \quad \lim_{n\to\infty} t_n = \sum_{k=1}^{\infty} \left[\frac{1}{k} - \log\left(\frac{k+1}{k}\right)\right]$$

The series above converges to a positive constant since

$$0 < \frac{1}{k} - \log\left(\frac{k+1}{k}\right) = \frac{1}{2k^2} - \frac{1}{3k^3} + \frac{1}{4k^4} - + \cdots \leq \frac{1}{2k^2}$$

This proves the lemma, because $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} t_n$. $\square$

Theorem (The Euler Gamma Function) There exists a unique function Γ on $\mathbb{C}$ such that

(a) Γ is meromorphic on $\mathbb{C}$

(b) $\forall\, n \in \mathbb{N}$, $\Gamma(n+1) = n!$

(c) $\Gamma(1/2) = \sqrt{\pi}$

(d) $\forall\, s \in \mathbb{C}$ such that $Re\,(s) > 0$

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1}\, dx$$

(e) $\forall\, s \in \mathbb{C}$ except for poles

$$\Gamma(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+s)} + \int_1^{\infty} e^{-x} x^{s-1}\, dx \quad \text{is meromorphic on } \mathbb{C}$$

(f) $\forall\, s \in \mathbb{C}$

$$\frac{1}{\Gamma(s)} = se^{\gamma s}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}} \quad \text{is entire, where } \gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log\, n\right)$$

is called the Euler constant (and note that $1/\Gamma(0)=0$).

(g) $\forall\, s \in \mathbb{C}$ except for poles

$$\Gamma(s) = \lim_{n\to\infty}\frac{n!\, n^s}{s(s+1)\cdots(s+n)}$$

(h) Γ has no zeros; in other words, $1/\Gamma$ is an entire function

(i) The poles of Γ are the nonpositive integers $s = 0, -1, -2, \ldots$. The pole of Γ at $s=-n$, with $n \in \mathbb{N}\cup\{0\}$ is a simple pole, with residue

$$\mathrm{Res}_{s=-n}\,\Gamma(s) = \frac{(-1)^n}{n!}$$

(j) $\forall\, s \in \mathbb{C}$ except for poles, $\Gamma(s+1) = s\Gamma(s)$

(k) $\forall\, s \in \mathbb{C}$ except for poles, $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$

Proof of (d) For each $s = \xi + i\eta \in \mathbb{C}$ such that $Re(s) > 0$, and for each $n \in \mathbb{N}$, let $\Gamma(s)$ and $f_n(s)$ be respectively defined by

$$\Gamma(s) = \int_0^\infty e^{-x}\, x^{s-1}\, dx \quad \text{and} \quad f_n(s) = \int_0^n e^{-x}\, x^{s-1}\, dx \quad \text{for } Re(s) > 0$$

Since

- f_n is analytic in $\{s \in \mathbb{C} \mid Re(s) > 0\}$ by Morera's Theorem,
- $|x^{s-1}| = x^{Re(s)-1}$ for $x > 0$,

$$|\Gamma(s) - f_n(s)| \le \int_n^\infty e^{-x}\, x^{Re(s)-1}\, dx$$

- $\displaystyle\int_n^\infty e^{-x}\, x^{Re(s)-1}\, dx$ converges uniformly in every strip $\{s \in \mathbb{C} \mid a \ge Re(s) \ge \delta > 0\}$,

so, by the Weierstrass Theorem, $\Gamma(s)$ is analytic on $\{s \in \mathbb{C} \mid Re(s) > 0\}$ with

$$\Gamma(1) = \int_0^\infty e^{-x}\, dx = -e^{-x}\Big|_0^\infty = 1,$$

and a singularity at $z = 0$ since

$$\lim_{\varepsilon\to 0^+}\Gamma(\varepsilon) = \lim_{\varepsilon\to 0^+}\int_0^\infty \frac{e^{-x}}{x^{1-\varepsilon}}\, dx \ge \lim_{\varepsilon\to 0^+}\int_0^1 \frac{e^{-x}}{x^{1-\varepsilon}}\, dx \ge \lim_{\varepsilon\to 0^+}\left.\frac{e^{-1}\, x^{\varepsilon}}{\varepsilon}\right|_0^1 = \infty \qquad \square$$

Proof of (b), (j) For s such that $Re\,(s) > 0$, integration by parts yields

$$\Gamma(s+1) = \int_0^\infty e^{-x}\, x^s\, dx = -e^{-x}\, x^s\Big|_0^\infty + s\int_0^\infty e^{-x}\, x^{s-1}\, dx = s\,\Gamma(s)$$

$$\Longrightarrow \Gamma(n+1) = n!\,\Gamma(1) = n!\int_0^\infty e^{-x}\, dx = n!\; -e^{-x}\Big|_0^\infty = n!\cdot 1 = n! \quad \forall\, n \in \mathbb{N} \quad \square$$

Proof of (c) When $s = 1/2$,

$$\Gamma(1/2) = \int_0^\infty e^{-x}\, x^{-1/2}\, dx \overset{t=\sqrt{x}}{=} 2\int_0^\infty e^{-t^2}\, dt = \sqrt{\pi}. \quad \square$$

Proof of (a), (i) We can use the functional equation $\Gamma(s+1) = s\Gamma(s)$ to analytically continue Γ to a meromorphic function on $\mathbb{C}$. Let

$$\Gamma_1(s) := \frac{\Gamma(s+1)}{s} \quad \text{for } Re(s) > -1,\ s \neq 0$$

Then

- Γ_1 is an extension of Γ on $Re(s) > -1$ since $\Gamma_1(s) = \Gamma(s)$ for $Re(s) > 0$,
- Γ_1 is analytic for $-1 < Re(s) < 0$,
- Γ_1 is continuous at each $s = i\eta$, $\eta \neq 0$, since

$$|\Gamma(i\eta+1)| \leq \int_0^\infty e^{-x}\, |x^{i\eta}|\, dx = \int_0^\infty e^{-x}\, dx = 1,$$
$$\text{and} \quad \lim_{s\to i\eta} \Gamma_1(s) = \lim_{z\to i\eta} \frac{\Gamma(z+1)}{z} = \frac{\Gamma(i\eta+1)}{i\eta} = \Gamma_1(i\eta), \ \text{ for } \eta \neq 0.$$

Hence, Γ_1 is analytic throughout $Re(s) > -1$, $s \neq 0$ by Morera's Theorem, coincides with Γ for $Re\,(s) > 0$, and has a simple pole at $s = 0$ with residue 1 since

$$\Gamma_1(s) = \frac{\Gamma(s+1)}{s} \sim \frac{\Gamma(1)}{s} = \frac{1}{s} \quad \text{as } s \to 0 \implies \mathrm{Res}_{s=0}\Gamma_1 = \lim_{s\to 0} s\Gamma_1(s) = \Gamma(1) = 1.$$

Likewise, for s such that $Re(s) > -2$, and $s \neq 0$, $s \neq -1$, let

$$\Gamma_2(s) := \frac{\Gamma_1(s+1)}{s} = \frac{\Gamma(s+2)}{s(s+1)} \quad \text{for } Re(s) > -2,\ s \neq 0,\ -1$$

Then Γ_2 is analytic on $\{s \in \mathbb{C} \mid Re(s) > -2\} \setminus \{0, -1\}$, coincides with Γ_1 for $Re\,(s) > -1$, and has a simple pole at $s = -1$ with residue -1 since

$$\mathrm{Res}_{s=-1}\Gamma_2 = \lim_{s\to -1}\,(s+1)\,\Gamma_2(s) = \lim_{s\to -1}\,(s+1)\,\frac{\Gamma(s+2)}{s(s+1)} = -1$$

Suppose Γ_{n-1} is the analytic continuation of Γ to $\{s \in \mathbb{C} \mid Re(s) > -2\} \setminus \{-k\}_{k=0}^{n-2}$. Let

$$\Gamma_n(s) := \frac{\Gamma_{n-1}(s+1)}{s} = \frac{\Gamma(s+n)}{s(s+1)\cdots(s+n-1)} \quad \text{for } Re(s) > -n,\ s \neq -k,\ 0 \leq k \leq n-1.$$

Then Γ_n is meromorphic on $Re(s) > -n$, and has a simple pole at $s = -k$ with residue $\dfrac{(-1)^k}{k!}$ for each $0 \leq k \leq (n-1)$, and the function Γ defined by

$$\Gamma(s) := \Gamma_n(s) \quad \text{for all } s \in \mathbb{C} \text{ such that } Re(s) > -n,\ s \neq -k$$

for each $n \in \mathbb{N}$, $0 \leq k \leq n-1$ is meromorphic on $\mathbb{C}$. $\square$

Proof of (e) Let us write

$$\Gamma(s) = \int_0^1 e^{-x}x^{s-1}\,dx + \int_1^\infty e^{-x}x^{s-1}\,dx$$

The second term on the right-hand side is analytic for all $s \in \mathbb{C}$, and the first term becomes

$$\int_0^1 e^{-x}x^{s-1}\,dx = \int_0^1 \sum_{n=0}^\infty \frac{(-1)^n}{n!}x^{n+s-1}\,dx = \sum_{n=0}^\infty \frac{(-1)^n}{n!}\int_0^1 x^{n+s-1}\,dx$$

where we have interchanged the order of summation and integration using absolute convergence. We thus get the final expression

$$\Gamma(s) = \sum_{n=0}^\infty \frac{(-1)^n}{n!(n+s)} + \int_1^\infty e^{-x}x^{s-1}\,dx \quad \text{is meromorphic on } \mathbb{C} \quad \square$$

Proof of (f) − (h) Consider $\lim_{n\to\infty} \int_0^n \left(1-\frac{x}{n}\right)^n x^{s-1}\,dx$ for $Re(s) > 0$.

Since

$$\lim_{n\to\infty} \left(1-\frac{x}{n}\right)^n x^{s-1} = e^{-x}x^{s-1}, \text{ and } \left(1-\frac{x}{n}\right)^n \le e^{-x} \quad \forall n \in \mathbb{N},\ \forall x \in [0,\, n],$$

hence, by the dominated convergence theorem,

$$\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}\,dx = \lim_{n\to\infty}\int_0^n \left(1-\frac{x}{n}\right)^n x^{s-1}\,dx$$

Now, for $Re(s) > 0$, we

Claim : $$\int_0^n \left(1-\frac{x}{n}\right)^n x^{s-1}\,dx = \frac{n!\,n^s}{s(s+1)\cdots(s+n)} \quad \forall n \in \mathbb{N}$$

Proof of Claim Since

$$\int_0^1 (1-x)\, x^{s-1}\,dx = \frac{1}{s(s+1)},$$

the **Claim** holds for $n = 1$. Assume it holds for $n-1$. Then

$$\begin{aligned}
&\int_0^n \left(1-\frac{x}{n}\right)^n x^{s-1}\,dx \\
&\overset{t=x/n}{=} n^s \int_0^1 (1-t)^n\, t^{s-1}\,dt = \frac{n^s}{s}\left\{[(1-t)^n t^s]_0^1 + n\int_0^1 (1-t)^{n-1}\, t^s\,dt\right\} \\
&= \frac{n^{s+1}}{s}\int_0^1 (1-t)^{n-1}\, t^s\,dt = n^s\cdot\frac{n}{s}\int_0^1 (1-t)^{n-1}\, t^{(s+1)-1}\,dt \\
&= n^s\cdot\frac{n(n-1)}{s(s+1)}\int_0^1 (1-t)^{n-2}\, t^{(s+2)-1}\,dt = \cdots = \frac{n!\,n^s}{s(s+1)\cdots(s+n)}
\end{aligned}$$

where we have used the induction hypothesis for the last step. Hence, for $Re(s) > 0$,

$$\Gamma(s) = \lim_{n\to\infty}\frac{n!\,n^s}{s(s+1)\cdots(s+n)}$$

We want to extend this result to $\mathbb{C}$, excluding the poles of Γ. Let us consider the function $1/\Gamma$. For $Re(s) > 0$,

$$\frac{1}{\Gamma(s)} = \lim_{n\to\infty} \frac{s(s+1)\cdots(s+n)}{n!\, n^s} = s \lim_{n\to\infty} e^{-s\log n}\,(1+s)\left(1+\frac{s}{2}\right)\cdots\left(1+\frac{s}{n}\right)$$
$$= s \lim_{n\to\infty} e^{s(\sum_{k=1}^{n} 1/k - \log n)} \prod_{k=1}^{n}\left(1+\frac{s}{k}\right) e^{-s/k} = se^{\gamma s}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right) e^{-s/n}$$

From the Weierstrass factorization theorem, we know that this represents an entire function with zeros at the nonpositive integers, which proves (f) – (h). $\square$

Proof of (k) For all $s \in \mathbb{C}$ except for the poles of Γ, one can write

$$\frac{1}{\Gamma(s)\Gamma(1-s)} = -\frac{1}{s\Gamma(s)\Gamma(-s)} = -\frac{1}{s} se^{\gamma s}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right) e^{-\frac{s}{n}}(-s)e^{-\gamma s}\prod_{n=1}^{\infty}\left(1-\frac{s}{n}\right) e^{\frac{s}{n}}$$
$$= s\prod_{n=1}^{\infty}\left(1-\frac{s^2}{n^2}\right) = \frac{\sin \pi s}{\pi} \qquad \square$$

Example (Volume of an n-dimensional Ball) Consider the function of n real variables

$$f(x_1, x_2, \ldots, x_n) = \exp\left(-\sum_{k=1}^{n} x_k^2/2\right)$$

We can evaluate

$$\int_{\mathbb{R}^n} f\, d\mathbf{x} = \prod_{k=1}^{n}\left(\int_{-\infty}^{\infty} e^{-x_k^2/2}\, dx_k\right) = (\sqrt{2\pi})^n \tag{14}$$

Now, since f is rotationally symmetric, one can use generalized spherical coordinates to rewrite the integral as follows:

$$\int_{\mathbb{R}^n} f\, d\mathbf{x} = \int_0^{\infty} e^{-r^2/2} \int_{S^{n-1}(r)} dA\, dr = \int_0^{\infty} e^{-r^2/2} A_{n-1}(r)\, dr$$

where $S^{n-1}(r)$ is the $(n-1)$-sphere of radius r, dA is the area element, and $A_{n-1}(r)$ is the surface area of the sphere $S^{n-1}(r)$.

Now, $A_{n-1}(r) = r^{n-1}A_{n-1}(1)$, so

$$\int_{\mathbb{R}^n} f\, d\mathbf{x} = A_{n-1}(1)\int_0^{\infty} r^{n-1}e^{-r^2/2}\, dr \overset{t=r^2/2}{=} 2^{(n-2)/2}A_{n-1}(1)\int_0^{\infty} t^{(n/2)-1}e^{-t}\, dt$$
$$= 2^{(n-2)/2}A_{n-1}(1)\Gamma\left(n/2\right) \tag{15}$$

Comparing Eq.(14) and (15), we obtain the equality:

$$A_{n-1}(r) = \frac{2\pi^{n/2}}{\Gamma\left(n/2\right)} r^{n-1}$$

Hence, $V_n(r)$, the volume of the n-ball of radius r is given by

$$V_n(r) = \int_0^{r} A_{n-1}(t)\, dt = \frac{2\pi^{n/2}}{\Gamma\left(n/2\right)}\int_0^{r} t^{n-1}\, dt = \frac{2\pi^{n/2}}{n\Gamma\left(n/2\right)} r^n = \frac{2\pi^{n/2}}{\Gamma\left((n/2)+1\right)} r^n \quad \square \tag{16}$$

§ 4 The Riemann Zeta Function

Just like the gamma function, the Riemann zeta function plays a key role in many fields of mathematics. It is however much less well understood and characterized than the zeta function. There remains several open problems associated with it, including the **Riemann hypothesis.**

Theorem (The Riemann Zeta Function) There exists a unique function ζ on $\mathbb{C}$ such that

(a) ζ is meromorphic on $\mathbb{C}$

(b) For $Re(s) > 1$,
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(c) For $Re(s) > 1$,
$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \quad \text{(called the Euler product formula)}$$
where, as indicated, the product ranges over the prime numbers.

(d) ζ has no zeros in the region $Re(s) > 1$

(e) ζ has no zeros on the line $Re(s) = 1$

(f) The zeros of ζ in the region $Re(s) \leq 0$ are at $s = -2k$, $k \in \mathbb{N}$

(g) ζ has a unique pole, at $s = 1$, with residue 1.

(h) The values of ζ at even positive integers are given by Euler's formula:
$$\zeta(2n) = \frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!} B_{2n} \ , \ n \in \mathbb{N}$$
where the B_k are the Bernoulli numbers, defined by the following Taylor expansion:
$$\frac{z}{e^z - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} z^m$$

(i) ζ takes the following values for negative integers:
$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \ , \ n \in \mathbb{N}$$

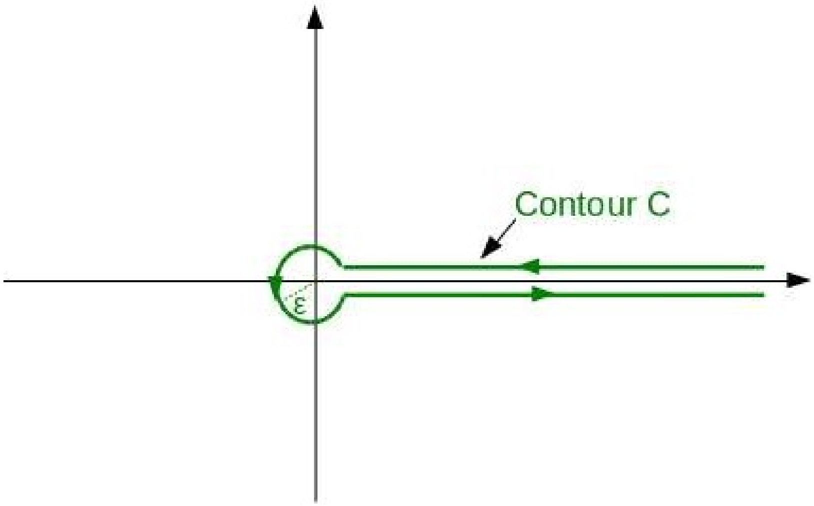
(j) ζ verifies the functional equation
$$\Lambda(1-s) = \Lambda(s)$$
where Λ is the symmetrized zeta function defined by
$$\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\,\zeta(s)$$

(k) $\forall s \in \mathbb{C} \setminus \{0, 1\}$
$$\pi^{-s/2}\Gamma(s/2)\,\zeta(s) = -\frac{1}{1-s} - \frac{1}{s} + \frac{1}{2}\int_1^{\infty} \left(t^{-(s+1)/2} + t^{(s-2)/2}\right)(\theta(t) - 1)\, dt$$
where the function θ is one of the Jacobi theta series, defined as
$$\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 t}$$

(l) $\forall\, s \in \mathbb{C} \setminus \{1\}$,

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^s}{e^z - 1} \frac{dz}{z}$$

where C is the keyhole contour shown in the following figure, with ε arbitrary as long as the circle does not enclose an integer multiple of $2\pi i$. The branch of the logarithm in the integrand is to be chosen such that $-\pi < \operatorname{Arg}(-z) < \pi$.



(m) *Connection to prime number enumeration*

Define $\psi(x) = \sum\limits_{p^k \le x} \ln p$, with p prime numbers.

ψ is called the *Von Mangoldt weighted prime counting function*. Then, for any noninteger $x > 1$,

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \ln 2\pi$$

where the sum is over the zeros ρ of the Riemann zeta function.

The formula (in (m)) above has important consequences for prime number enumeration, provided one can locate the zeros ρ of ζ in the complex plane. For example, the fact that ζ has no zeros such that $Re(s) \ge 1$ leads, after some work, to the prime number theorem given below.

Theorem (Prime Number Theorem) Let $\pi(x)$ denote the number of prime numbers less than or equal to x. We have

$$\lim_{x \to \infty} \frac{\pi(x)}{(x/\log x)} = 1$$

Of course, the exact location of the nontrivial zeros of the Riemann zeta function remains a key open problem. It is usually described as the Riemann hypothesis, which conjectures that all the nontrivial zeros of ζ are on the line $Re(s) = 1/2$, called the *critical line*.

Proof of (b) Recall that the Zeta Function $\zeta(s)$ is defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots, \quad Re(s) > 1. \lim_{\varepsilon \to 0^+} \zeta(1+\varepsilon) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

Since $|n^s| = |n^{Re\, s}|$, the series $\sum_{n=1}^{\infty} 1/n^s$ converges absolutely for $Re(s) > 1$, uniformly on any half-plane $Re(s) > \delta$ with $\delta > 1$. Hence ζ is analytic on $Re(s) > 1$, and has a singularity at $s = 1$ since $\lim_{\varepsilon \to 0^+} \zeta(1+\varepsilon) = \sum_{n=1}^{\infty} 1/n = \infty$. $\square$

Proof of (c) Likewise

$$\prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

converges absolutely iff $\sum_{p \text{ prime}} |p^{-s}| = \sum_{p \text{ prime}} p^{-Re(s)}$ converges, which happens for $Re(s) > 1$. Hence

$$F(s) := \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

is analytic and nonzero in $Re(s) > 1$. It remains to show that $\zeta(s) = F(s)$ on this set.
For $Re(s) > 1$, let

$$\zeta_N(s) := \prod_{p \le N, \text{ prime}} \frac{1}{1-p^{-s}} \overset{|p^{-s}|=p^{-Re(s)}<1}{=} \prod_{p \le N} \sum_{k=0}^{\infty} p^{-ks} = \sum_{n=p_1^{c_1} \cdots p_m^{c_m}} \frac{1}{n^s} \quad p_i \le N, \text{ prime}, 1 \le i \le m$$

where the last equality was obtained by reorganizing terms in the absolutely convergent series. Hence, by the fundamental theorem of arithmetic,

$$\lim_{N\to\infty} |\zeta(s) - \zeta_N(s)| \le \lim_{N\to\infty} \sum_{n>N} \frac{1}{n^s} = 0$$

which proves that for $Re(s) > 1$,

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \quad \text{(called the Euler product formula)} \qquad \square$$

Proof of (d) follows immediately from the Euler product formula. $\square$

Alternative Proof of (c), (d) Since

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{and} \quad \frac{1}{2^s}\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{(2n)^s} \quad \text{for } Re(s) > 1$$

$$\Longrightarrow \left(1 - \frac{1}{2^s}\right)\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s}, \quad \text{and} \quad \frac{1}{3^s}\left(1 - \frac{1}{2^s}\right)\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{(6n-3)^s}$$

$$\Longrightarrow \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \cdots$$

and because of the unique prime factorization of the integers, we can continue indefinitely to obtain (in the limit)

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)\zeta(s) = 1 \Longrightarrow \zeta(s) = 1/\prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}, \quad Re(s) > 1$$

Note that

$$\begin{aligned}
\frac{\Gamma(s)}{n^s} &= \frac{1}{n^s}\int_0^\infty e^{-u}\,u^{s-1}\,du \overset{u=nt}{=} \int_0^\infty e^{-nt}\,t^{s-1}\,dt \quad \text{for each } n\in\mathbb{N}\\
\Longrightarrow \Gamma(s)\sum_{n=1}^\infty \frac{1}{n^s} &= \int_0^\infty t^{s-1}\left(\sum_{n=1}^\infty e^{-nt}\right)dt = \int_0^\infty \frac{t^{s-1}}{e^t-1}\,dt\\
\Longrightarrow \zeta(s) &= \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}}{e^t-1}\,dt = \frac{1}{\Gamma(s)}\left[\int_0^1 \frac{t^{s-1}}{e^t-1}\,dt + \int_1^\infty \frac{t^{s-1}}{e^t-1}\,dt\right]
\end{aligned}$$

Recall that $1/\Gamma(s)$ (with the appropriate limiting value of zero at the poles of $\Gamma(s)$) is entire, as is $\displaystyle\int_1^\infty \big(t^{s-1}/(e^t-1)\big)\,dt$. Furthermore, the Laurent Expansion for $1/(e^t-1)$ around $t=0$,

$$\frac{1}{e^t-1} = \frac{1}{t} + A_0 + A_1\,t + A_2\,t^2 + \cdots,$$

converges absolutely for $t=1$ so that

$$\int_0^1 \frac{t^{s-1}}{e^t-1}\,dt = \int_0^1 \left(t^{s-2} + A_0\,t^{s-1} + A_1\,t^s + \cdots\right)dt = \frac{1}{s-1} + \frac{A_0}{s} + \frac{A_1}{s+1} + \cdots$$

provides an analytic extension of $\displaystyle\int_0^1 \big(t^{s-1}/(e^t-1)\big)\,dt$ except for isolated poles. Thus

$$\zeta(s) = \frac{1}{\Gamma(s)}\left[\left(\frac{1}{s-1} + \frac{A_0}{s} + \frac{A_1}{s+1} + \cdots\right) + g(s)\right]$$

where $g(s)$ is entire. Note that while he bracketed expression above has a simple pole at $s=1$ as well as at every non-positive integer, all these poles are cancelled by the zeros of $1/\Gamma(s)$ except $s=1$. Hence ζ has a single (simple) pole at $s=1$ with residue 1. □

Proof of (k) For $t>0$, $x\in\mathbb{R}$, let $F(x) := \displaystyle\sum_{n\in\mathbb{Z}} \exp\left(-\pi t(n+x)^2\right)$. Since $\{e^{2\pi ikx} \mid k\in\mathbb{Z}\}$ forms an orthonormal basis for $L^2([0,1])$, and $F(x+1)=F(x)$ for all $x\in\mathbb{R}$, so F is a periodic function of period 1, and $F(x) = \displaystyle\sum_{k\in\mathbb{Z}} \widehat{f_n}(k)\,e^{2\pi ikx}$, where $f_n(x) = \exp\left(-\pi t(n+x)^2\right)$, and $\widehat{f_n}(k)$ is the Fourier coefficient defined by

$$\widehat{f_n}(k) := \int_{-\infty}^\infty f_n(x)e^{-2\pi ikx}\,dx = \frac{1}{\sqrt{t}}\exp\left(-\pi k^2/t\right) \quad \text{for each } k\in\mathbb{Z}$$

Since $\displaystyle\sum_{n\in\mathbb{Z}} e^{-\pi t(n+0)^2} = F(0) = \sum_{k\in\mathbb{Z}} \widehat{f_n}(k)\,e^{2\pi ik\cdot 0}$, the theta function $\theta(t) = \displaystyle\sum_{n=-\infty}^\infty e^{-\pi n^2 t}$ satisfies

$$\theta(t) = \sum_{n=-\infty}^\infty e^{-\pi n^2 t} = F(0) = \sum_{k\in\mathbb{Z}} \widehat{f_n}(k) = \sum_{k=-\infty}^\infty \frac{1}{\sqrt{t}}e^{-\frac{\pi k^2}{t}} = \frac{1}{\sqrt{t}}\,\theta\left(1/t\right)$$

We observe that for $t>0$,

$$\theta(t) - 1 = 2\sum_{n=1}^\infty e^{-\pi n^2 t} \le 2\sum_{n=1}^\infty e^{-\pi nt} = 2\frac{e^{-\pi t}}{1-e^{-\pi t}}$$

Thus, $\theta(t) = 1 + O(e^{-\pi t})$ for $t \to \infty$. So using the equality $\theta(t) = \frac{1}{\sqrt{t}}\theta\left(1/t\right)$, we conclude that

$$\theta(t) = \frac{1}{\sqrt{t}}\left(1 + O(e^{-\pi/t})\right) \ \text{as}\ t \to 0^+ \implies \theta(t) = O\left(1/\sqrt{t}\right) \ \text{as}\ t \to 0^+$$

We now turn to the Mellin transform representation for $\zeta(s)$ $Re(s) > 1$ by writing

$$\begin{aligned}
\Gamma\left(s/2\right) &= \int_0^\infty e^{-x}\, x^{\frac{s}{2}-1}\, dx \quad \text{setting } x = \pi n^2 t \\
\iff \pi^{-s/2}\,\Gamma\left(s/2\right) n^{-s} &= \int_0^\infty e^{-\pi n^2 t}\, t^{\frac{s}{2}-1}\, dt \quad \text{summing over } n \in \mathbb{N} \\
\implies \pi^{-s/2}\,\Gamma\left(s/2\right)\,\zeta(s) &= \sum_{n=1}^{\infty} \pi^{-s/2}\,\Gamma\left(s/2\right) n^{-s} = \Lambda(s) = \sum_{n=1}^{\infty} \int_0^\infty e^{-\pi n^2 t}\, t^{\frac{s}{2}-1}\, dt
\end{aligned}$$

where

$$\Lambda(s) = \sum_{n=1}^{\infty} \int_0^\infty e^{-\pi n^2 t}\, t^{\frac{s}{2}-1}\, dt = \int_0^\infty \left(\sum_{n=1}^{\infty} e^{-\pi n^2 t}\right) t^{\frac{s}{2}-1} dt = \int_0^\infty \frac{\theta(t)-1}{2}\, t^{\frac{s}{2}-1}\, dt$$

by using the estimates for θ to exchange the sum and integral signs. Now, let

$$g(t) := \frac{\theta(t)-1}{2}$$

Since g satisfies the equality

$$g(t) = \frac{\theta(t)-1}{2} = \frac{1}{\sqrt{t}}\frac{\theta(1/t)-1}{2} + \frac{1}{2\sqrt{t}} - \frac{1}{2} = \frac{1}{\sqrt{t}} g\left(1/t\right) + \frac{1}{2\sqrt{t}} - \frac{1}{2}$$

we have

$$\begin{aligned}
\Lambda(s) &= \int_0^1 g(t)\, t^{\frac{s}{2}-1}\, dt + \int_1^\infty g(t)\, t^{\frac{s}{2}-1}\, dt = \int_0^1 \left(\frac{1}{\sqrt{t}} g\left(1/t\right) + \frac{1}{2\sqrt{t}} - \frac{1}{2}\right) t^{\frac{s}{2}-1}\, dt + \int_1^\infty g(t)\, t^{\frac{s}{2}-1}\, dt \\
&\overset{u=1/t}{=} -\frac{1}{s} - \frac{1}{1-s} + \int_1^\infty g(u)\, u^{-\frac{s}{2}-\frac{1}{2}}\, du + \int_1^\infty g(t)\, t^{\frac{s}{2}-1}\, dt \\
&= -\frac{1}{s} - \frac{1}{1-s} + \frac{1}{2}\int_1^\infty \left(\theta(t)-1\right)\left(t^{-\frac{s}{2}-\frac{1}{2}} + t^{\frac{s}{2}-1}\right) dt \qquad \square
\end{aligned}$$

Proof of (a) The integral defines an entire function on $\mathbb{C}$, showing that ζ can indeed be continued to a meromorphic function on $\mathbb{C}$. $\square$

Proof of (j) The Mellin transform representation immediately yields $\Lambda(1-s) = \Lambda(s)$, which is property (j), and which can be rewritten as

$$\Lambda(s) = 2^s \pi^{s-1} \sin\left(\pi s/2\right)\Gamma\left(1-s\right)\zeta(1-s) \tag{17}$$

Proof of (g) Λ has simple poles at $s = 0$ and at $s = 1$, with residues -1 and 1. Therefore, $\zeta(s) = \frac{s^{s/2}}{\Gamma\left(s/2\right)}\Lambda(s)$ has a pole at $s = 1$ with residue $\sqrt{\pi}/\Gamma\left(1/2\right) = 1$ and a pole at $s = 0$ with residue $1/\Gamma(0) = 0$ We see that the singularity at 0 is in fact a removable singularity. $\square$

Proof of (f) By the Eq.(17), it is clear that the zeros of ζ for $Re(s) < 0$ are precisely $s = -2k$, $k \in \mathbb{N}$. $\square$

Alternative Proof of (e) Let $\sigma > 1$ and $t \in \mathbb{R}^*$ and consider the quantity

$$
\begin{aligned}
\mu &= \ln\left|\zeta(\sigma)^3\,\zeta(\sigma+it)^4\,\zeta(\sigma+2it)\right| = 3\ln|\zeta(\sigma)| + 4\ln|\zeta(\sigma+it)| + \ln|\zeta(\sigma+2it)| \\
&= 3\ln\left|\prod_{p\text{ prime}} \frac{1}{1-p^{-\sigma}}\right| + 4\ln\left|\prod_{p\text{ prime}} \frac{1}{1-p^{-\sigma-it}}\right| + \ln\left|\prod_{p\text{ prime}} \frac{1}{1-p^{-\sigma-2it}}\right| \\
&= \sum_{p\text{ prime}} \left(-3\ln|1-p^{-\sigma}| - 4\ln|1-p^{-\sigma-it}| - \ln|1-p^{-\sigma-2it}|\right) \\
&= \sum_{p\text{ prime}} \left[-3\,Re\left(\mathrm{Ln}\,(1-p^{-\sigma})\right) - 4\,Re\left(\mathrm{Ln}\,(1-p^{-\sigma-it})\right) - Re\left(\mathrm{Ln}\,(1-p^{-\sigma-2it})\right)\right]
\end{aligned}
$$

where, as always, Ln is the principal branch of the logarithm. Our next step will be to use power series for Ln, which we can since

$$|p^{-\sigma}| < 1 \ , \ |p^{-\sigma-it}| < 1 \ , \ |p^{-\sigma-2it}| < 1$$

For $s = a + ib$ such that $a = Re(s) > 1$,

$$-\mathrm{Ln}\,(1-p^{-s}) = \sum_{k=1}^{\infty} \frac{p^{-ks}}{k} \implies -Re\left(\mathrm{Ln}\,(1-p^{-s})\right) = \sum_{k=1}^{\infty} \frac{p^{-ks}}{k}\,\cos(kb\ln p)$$

Therefore,

$$\mu = \sum_{p\text{ prime}} \sum_{k=1}^{\infty} \frac{p^{-ks}}{k}\left[3 + 4\cos(kt\ln p) + \cos(2kt\ln p)\right] = 2\sum_{p\text{ prime}} \sum_{k=1}^{\infty} \frac{p^{-ks}}{k}\left[1+\cos(kt\ln p)\right]^2 \geq 0$$

We can therefore say that $e^{\mu} \geq 1$, which means that

$$\left|\zeta(\sigma)^3\,\zeta(\sigma+it)^4\,\zeta(\sigma+2it)\right| = e^{\mu} \geq 1 \tag{18}$$

All we now have to show is that this inequality prevents ζ from having a zero on the line $Re(s) = 1$. Let us assume the contrary: $\exists\, t \in \mathbb{R}^*$ such that $\zeta(1+it) = 0$. We then look at the asymptotic behavior of each term in (18) as $\sigma \to 1^+$:

$$\zeta(\sigma) \sim \frac{1}{\sigma-1} \ ; \ \zeta(\sigma+it) \sim K_1(\sigma-1) \ ; \ \zeta(\sigma+2it) \sim K_2 \quad \text{as } \sigma \to 1^+ \ , \ K_1, K_2 \in \mathbb{C}$$

where the first asymptotic estimate is tight, and the other two are conservative, in the sense that $\zeta(\sigma+it)$ could go to zero faster, and $\zeta(\sigma+2it)$ could also go to zero as $\sigma \to 1^+$. We then obtain the following conservative asymptotic estimate as $\sigma \to 1^+$:

$$\left|\zeta(\sigma)^3\,\zeta(\sigma+it)^4\,\zeta(\sigma+2it)\right| \sim K_3(\sigma-1) \quad \text{as } \sigma \to 1^+ \ , \ K_3 \in \mathbb{C}$$

This contradicts the result $|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)| \geq 1$ for all $\sigma > 1$ and $t \in \mathbb{R}^*$. ζ does not have any zero with real part equal to 1. $\square$

Alternative Proof of (e) Since $\zeta(s)$ is real-valued for real s, according to the Schwarz Reflection Principle, $\zeta(\overline{s}) = \overline{\zeta(s)}$ for all $s \in \mathbb{C}$.

Thus, if $\zeta(1+ia) = 0$ for some $a \in \mathbb{R}$, then $\zeta(1-ia) = 0$, the functions $f(s) = \zeta(s)\,\zeta(s+ia)$, $f(s-ia) = \zeta(s-ia)\,\zeta(s)$ and $g(s) = f(s)\,f(s-ia) = \zeta^2(s)\,\zeta(s+ia)\,\zeta(s-ia)$ are entire since the pole of $\zeta(s)$ is cancelled by the zero $\zeta(s+ia)$ at $s=1$, and the pole of $\zeta(s+ia)$ is cancelled by the zero of $\zeta(s)$ at $s=1-ia$.

The desired contradiction will be based, in part, on the fact that the Dirichlet series for $g(s)$ has all nonnegative coefficients. To see that, we first consider $\log(g(s))$ which, according to Euler's formula for $\zeta(s)$, is given by

$$\begin{aligned}\log(g(s)) &= \sum_{p \text{ prime}} \left[-2\log\left(1-p^{-s}\right) - \log\left(1-p^{-s+ia}\right) - \log\left(1-p^{-s-ia}\right)\right] \\ &= \sum_{p,n} \frac{1}{n\,p^{n\,s}}\left(2+p^{-i\,n\,a}+p^{i\,n\,a}\right) = \sum_{p,n} \frac{2+2\cos\left(n\,a\log p\right)}{n\,p^{n\,s}}\end{aligned}$$

where the sum is taken over all primes p and all positive integers n. Since $2+p^{-i\,n\,a}+p^{i\,n\,a} = 2+2\cos\left(n\,a\log p\right) \geq 0$, all of the coefficients in the above Dirichlet series for $\log(g(s))$ are nonnegative. But if a Dirichlet series $z(s) = \sum_n a_n n^{-s}$ has all nonnegative coefficients, so does

$$e^{z(s)} = \prod_n \sum_k \frac{a_n^k}{n^{k\,s}\,k\,!}$$

Hence $g(s)$ represents an entire function whose Dirichlet series has all nonnegative coefficients which implies that its Dirichlet series must converge for all $s \in \mathbb{C}$. But this is clearly impossible. Since all of the coefficients of

$$g(s) = \left(\sum_n n^{-s}\right)^2 \sum_n n^{-s-ia} \sum_n n^{-s+ia}$$

are nonnegative, the sum is clearly positive for all real s. Moreover, the sum must be larger than the sum over any subset of the positive integers. So if we consider nonnegative real values of s and limit ourselves to the subseries corresponding to integers n of the form 2^k, we have

$$|g(s)| > \frac{1}{(1-2^{-s})^2} \cdot \frac{1}{1-2^{-s-ia}} \cdot \frac{1}{1-2^{-s+ia}}$$

Finally, since s is nonnegative, $\left|\left(1-2^{-s-ia}\right)\left(1-2^{-s+ia}\right)\right| \leq 4$, and, by letting $s \to 0$ through positive real values, we have

$$\lim_{s\to 0}|g(s)| \geq \lim_{s\to 0}\frac{1}{4\left(1-2^{-s}\right)^2} = \infty$$

contradicts to that the Dirichlet series for $g(s)$ converges for all $s \in \mathbb{C}$. $\square$.

Proof of (1) Let $s \in \mathbb{C}$ such that $Re\,(s) > 1$. Since

$$\int_0^\infty \frac{t^{s-1}}{e^t-1}\,dt = \int_0^\infty t^{s-1}\sum_{n=1}^\infty e^{-nt}\,dt = \sum_{n=1}^\infty \int_0^\infty t^{s-1}e^{-nt}\,dt \overset{u=nt}{=} \sum_{n=1}^\infty \frac{1}{n^s}\int_0^\infty u^{s-1}e^{-u}\,du = \zeta(s)\Gamma(s)$$

where we have used absolute convergence to interchange the order of integration and summation, and note that if the curve C does not pass through or enclose a pole of the integrand, then, by Cauchy's Theorem, the value of the integral

$$\int_C \frac{(-z)^s}{e^z-1}\frac{dz}{z}$$

is a multiple of $2\pi i$ which does not depend on the shape of the curve C. We are therefore free to choose the key hole contour shown in the figure, and to take the limit $\varepsilon \to 0$ for that contour. It is then straightforward to verify that the contribution from the circle of radius ε tends to zero. When $\varepsilon \to 0$, the only contributions to the integral thus come from the two extended branches of the contour C and we have

$$\begin{aligned}
&\int_C \frac{(-z)^s}{e^z - 1} \frac{dz}{z} \\
&= \int_\infty^0 \frac{\rho^{s-1} e^{-is\pi}}{e^\rho - 1} \, d\rho + \int_0^\infty \frac{\rho^{s-1} e^{is\pi}}{e^\rho - 1} \, d\rho \\
&= (e^{is\pi} - e^{-is\pi}) \int_0^\infty \frac{\rho^{s-1}}{e^\rho - 1} \, d\rho = 2i \sin(s\pi) \, \zeta(s) \Gamma(s)
\end{aligned}$$

It is a simple exercise to verify that the integral over the small circle tends to zero as ε tends to 0 when $Re\,(s) > 1$. Hence, for $Re\,(s) > 1$,

$$\begin{aligned}
&\int_C \frac{(-z)^s}{e^z - 1} \frac{dz}{z} = 2i \sin(s\pi) \, \zeta(s) \Gamma(s) \\
\iff &\zeta(s) = \frac{1}{2i \sin(s\pi) \, \Gamma(s)} \int_C \frac{(-z)^s}{e^z - 1} \frac{dz}{z} = \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^s}{e^z - 1} \frac{dz}{z} \qquad (19)
\end{aligned}$$

We observe that the integral in (19) is an entire function of s, so (19) can be viewed as a way to analytically extend ζ to a meromorphic function in $\mathbb{C}$ which is equivalent to the Mellin transform representation. Note that when $Re\,(s) \leq 1$, it is not true anymore that the contribution from the circle of radius ε tends to 0 as $\varepsilon \to 0$. □

Proof of (i) Property (i) follows from (19). This is left as a straightforward exercise, as well as property (h) which follows from property (i) and the functional equation. □